EXCELLENCIA
INTERNATIONAL MULTI-DISCIPLINARY JOURNAL OF EDUCATION

Volume 02, Issue 06, 2024 ISSN (E): 2994-9521

On One Method Approximation of Diffusion Problems

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Abstract

The paper describes methods for approximating functionals from diffusions and from an optimally controlled diffusion process, as well as methods for approximating diffusion processes that are solutions of stochastic differential equations of Ito, both controlled and uncontrolled. Since many of the functionals that we will calculate and approximate are in fact weak solutions of partial differential equations (the weak solution can be represented as some functional of a suitable diffusion process), the methods for approximating weak solutions are closely related to the methods for approximating diffusion processes and their functionals. In addition, the appearance of partial differential equations, which, at least formally, satisfy the functionals we are interested in, suggests numerical methods for solving these problems.

Key words: bounded area, parabolic equation, parabolic equation, Cauchy problem, Wiener process, Markov processes.

Let G be a bounded area in R^n , $Q=G*[T_0.T_1]$ and a cylinder in R^{n+1}. Let's denote $\Gamma=Q\setminus Q$. Consider the first boundary value problem for a parabolic equation:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}\left(t,x\right)\frac{\partial^2 u}{\partial x^i\partial x^j}+\sum_{i=1}^{n}b^i\left(t,x\right)\frac{\partial u}{\partial x^i}+c\left(t,x\right)u+g\left(t,x\right)u=0\ . \tag{1.1}$$

$(t,x)\in Q, \quad u|r=\varphi(t,x)$ (1.2)

It is assumed that the coefficients satisfy the condition of strict ellipticity in $\overline{Q}$. In addition, we assume that the conditions that ensure sufficient smoothness of the solution of problem (1.1) up to the boundary are satisfied.

The solution of problem (1.1) admits a probabilistic representation

$$u(t,x)=E\left[\varphi(\tau,X_{t,x}(\tau))Y_{t,z,1}(\tau)+Z_{t,x,1,0}(\tau)\right], \tag{1.3}$$

where $X_{t,x}(S),\ Y_{t,x,y}(S),\ Z_{t,x,y,z}(S),\ S\geq t,$ is the solution of the Cauchy problem for a system of stochastic differential equations

$$dX=b(s,X)ds+\sigma(s,X)d\omega(s), \quad X(t)=x,$$

$$dY=c(s,X)Y\,ds, \quad Y(t)=y, \quad dZ=g(s,X)Y\,ds, \quad Z(t)=z, \tag{1.4}$$

$(t,x)\in Q,\tau$ -the moment when the trajectory $(s,X_{t,x}(s))$ reaches the boundary of Γ.

In $\omega(a)=(\omega^1(a))\ ,...,\ (\omega^n(a))^T$ -standard Wiener process, Y and Z - scalars $b(s,x)$ - n - a dimensional column vector made up of coefficients $b^i(s,x),\ n\times n$ - the matrix $\sigma(s,x)$ is obtained from the representation $\sigma(s,x)\sigma^T(s,x)=a(s,x)$, $a(s,x)=\{a^{ij}(s,x)\},i,j=\overline{1,n}$. To implement the representation (1.3), we need an approximate construction of the trajectory $(s,X(s))$.

It is known that the solution of the general Dirichlet problem is related in a certain way to a system of stochastic differential equations. Using some approximation methods, we can construct a Markov chain with absorption, which approximates the solutions of this system so that the mathematical expectation of a certain functional from the trajectories of the chain is close to the solution of boundary value problems for linear parabolic and elliptic equations of the second order. If a probabilistic representation of the Cauchy problem is used, then it is possible to construct direct approximations for solving the Cauchy problem, which are based on the constructed approximation method. However, if we want to approximate the solution of a parabolic or elliptic equation in a bounded domain in a similar way, then we need to have an approximation of the time of the first exit of diffusion through the boundaries of the domain.

Let $w(\bullet)$ r be a dimensional standard Wiener process, and $B_t=B(w(s),s\leq t)$. Non-proactive (relatively $w(\square)$) solutions to the equation

$$X(t)=x+\int f(X(s),s)ds+\int\sigma(X(s),s)dw(s) \tag{1.5}$$

Define a large class of Markov processes. Equation (1.5) is often written in symbolic differential form

$$dx=f(x,s)gs+\sigma(x,s)dw(s)$$

Such processes are widely used in stochastic control theory and other applications in engineering, physics, and economics, and describe many practically useful processes. Functionals from these processes are solutions of elliptic and parabolic partial differential equations, and therefore the study of the properties of such processes provides a lot of useful information about the properties of partial differential equations. In fact, the relations between the process (1.5) and partial differential equations will often be used in the study of process approximations and differential equations.

The computational methods that we will use are identical to the methods of computing functionals from finite Markov chains. We will obtain Markov processes of diffusion processes, roughly speaking, in the following way. Let's take a partial differential equation, which, at least formally, is satisfied by the functional of the diffusion process, and write the corresponding emu equation in finite differences. If the approximations are chosen carefully (but in an absolutely natural way), then the finite difference equation will actually turn out to be an equation for the functional of a Markov chain, and its transition probabilities will be coefficients of the finite difference equation.

When studying dynamical systems using computer technologies, the method of statistical tests (Monte Carlo method) is often used. The application of this method to the study of systems defined by stochastic differential equations requires their replacement with the Euler and Runge-Kutta difference schemes. Such substitutions are considered in the works. However, the known error estimates of difference methods for solving deterministic equations cannot be used in digital modeling of stochastic equations due to the non-differentiability of almost all their solutions. In this paper, a stochastic analog of the Taylor series is obtained for estimating errors, which allows us to decompose the solution of a stochastic differential equation into a series with respect to nonlinear functionals from the Wiener process.

The main attention in the works on numerical integration of stochastic differential equations is paid to the approximation of solutions in the root-mean-square sense. Meanwhile, in cases where the modeling of solutions is intended for the use of Monte Carlo methods, it is not necessary to solve a very complex problem of finding root-mean-square approximations. If $X(t)$-is exactly the solution, and $\bar{X}(t)$-is an approximate solution, then for many problems of mathematical physics it is only necessary that the mathematical expectation $Ef\left(\bar{X}(t)\right)$beat close to $Ef\left(\bar{X}(t)\right)$. i.e., that $\bar{X}(t)$ it beat close to $X(t)$ in the weak sense.

Of course, when [1] is numerically integrated in the root-mean-square sense with a certain order of accuracy, approximations in the weak sense are obtained with the same order of accuracy. Since if $E\left(\left|\bar{X}(t)-X(t)\right|^2\right)^{1/2}=0\ \left(h^p\right)$, then for any function f satisfying the Lipschitz condition , equality $E\left(f\left(\bar{X}(t)\right)-f\left(X(t)\right)\right)=0\ \left(h^p\right)$holds , however, as shown in modeling $\Delta_k w(h)=w(t_k+h)-w(t_k)$ (and even by modeling simpler random variables), it is possible to construct a second-order accuracy method in a weak sense. While in the root-mean-square sense, this method cannot be used to construct a method higher than the first order of accuracy. But even this is not the main factor that stimulates the development of methods for constructing weak approximations. It is known that even in root-mean-square systems with multiple noises, a difficult problem of modeling random variables of the form $\int_0^h w_i(v)\,dw_j(v)$ arises . This problem of modeling complex random variables can be avoided by integrating in a weak sense.

If we keep in mind the applications of the Monte Carlo method that show their effectiveness in multidimensional problems, then the development of numerical integration methods in the weak sense is very relevant just for systems with many noises.

References:

1. Tozhiev T., Ibragimov Sh., Rakhimov K. About modern modeling methods. Scientific and technical journal of FerPI. No. 1. 2018, 119 p.
2. Tozhiev T., Ibragimov Sh., Yusupaliev D. Monte Carlo method for solving the boundary value problem of the diffusion equation. Scientific and technical journal of FerPI. No. 4. 2017, 24 p.
3. Kuptsov L. On the property of the mean for a generalized Kolmogorov equation I / / Diff. equation 1983, vol. XIX, No. 2.
4. Тожиев Т. Х., Алдашев И. Т. Об одном методе аппроксимация диффузионных задач //Modern Scientific Research International Scientific Journal. – 2023. – Т. 1. – №. 3. – С. 11-17.
5. T Tozhiev, S Abdullaev - Creation of new numerical simulation algorithms for solving initial-boundary-value problems for diffusion equations//AIP Conference Proceedings, 2023
6. Komiljonovna M. M. et al. DEEP LEARNING: DEFINITION AND DISTINCTIVE FEATURES //IMRAS. – 2024. – Т. 7. – №. 1. – С. 198-202.
7. TT Halimovich, IS Mamirovich, KA Muminovich-Monte Carlo method for constructing an unbelised assessment of diffusion problems//European science review, 2020
8. TH Tojiev, SM Ibragimov - Numerical solutions of the cauchy problem for the generalized equation of nonisotropic diffusion//Bulletin of Namangan State University: Vol, 2019
9. Т Тожиев - ПРИМЕНЕНИЕ МЕТОДОВ МОНТЕ-КАРЛО ДЛЯ АППРОКСИМАЦИИ ДИФФУЗИОННЫХ ЗАДАЧ// International journal of scientific researchers (IJSR) …, 2024