EXCELLENCIA
INTERNATIONAL MULTI-DISCIPLINARY JOURNAL OF EDUCATION
Volume 02, Issue 10, 2024 ISSN (E): 2994-9521

Periodicity of the Discrete Dynamical Systems

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Annotation:

In the present paper, we consider cubic stochastic operators defined on a finite-dimensional simplex, which are depend on a permutation. We showed that for any permutation , except the identity permutation, the trajectories of the corresponding operators converges to a periodic trajectory. The trajectories of the operator corresponding to the identity permutation converge to a fixed point.

Keywords: Cubic stochastic operator, finite-dimensional simplex, permutation, fixed point.

The evolution of biological populations can be studied by dynamical systems of nonlinear operators. One of the simplest nonlinear cases is the quadratic one. Another class of such operators is the class of cubic operators (see e.g. [1-4]).

Let $E=\{1,2,\dots,m\}$ be a finite set. The set of all probability distribution on E is

$$S^{m-1}=\left\{\mathbf{x}=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m : x_i\ge 0, \text{ for any } i \text{ and } \sum_{i=1}^{m}x_i=1\right\},$$

which is called the $(m-1)$-dimensional simplex.

A cubic stochastic operator (CSO) is a mapping $W:S^{m-1}\to S^{m-1}$ of the form

$$W: x_l'=\sum_{i,j,k\in E}P_{ijk,l}x_ix_jx_k, \quad l\in E, \quad (1)$$

where $P_{ijk,l}$ are the coefficients such that

$$P_{ijk,l} \geq 0, \sum_{l=1}^{m} P_{ijk,l} = 1, \ \forall i,j,k \in E. \quad (2)$$

For a given $\mathbf{x}^{(0)} \in S^{m-1}$ the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$ of an initial point $\mathbf{x}^{(0)}$ under the action of W is defined by $\mathbf{x}^{(n+1)} = W(\mathbf{x}^{(n)})$, $n = 0,1,2,\dots$ We denote by $\omega_W(\mathbf{x}^{(0)})$ the ω-limit set of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0,1,2,\dots}$.

One of the main problems in mathematical biology consists of the study of the asymptotical behavior of the trajectories.

Definition 1. *A point $\mathbf{x} \in S^{m-1}$ is called a periodic point of W if there exists an n so that $W^n(\mathbf{x}) = \mathbf{x}$. The smallest positive integer n satisfying the above is called the prime period or least period of the point $\mathbf{x}$. A period-one point is called a fixed point of W. Denote the set of all fixed points by $\mathrm{Fix}(W)$ and the set of all periodic points of prime period n by $\mathrm{Per}_n(W)$.*

Consider CSO defined on S^{m-1} which has the form

$$W: \begin{cases} x'_k = x_{\pi(k)}\left(1+(c-1)x_m\right)^2, \ 1 \leq k \leq m-1, \\ x'_m = x_m\left(a+(1-a)x_m\right)\left(b+(1-b)x_m\right), \end{cases} \quad (3)$$

where π is a permutation of $E_{m-1} = \{1,2,\dots,m-1\}$ and

$$a+b+c^2 = 3, \ ab+2c = 3, \ a,b \in [0,1], \ c \in [1,3/2]. \quad (4)$$

Note that if $a=1$ that from (4) we get $b=c=1$. Clearly, if $a=1, \pi = \mathbf{Id}$ then the SCSO W (3) is identity map.

A permutation π of the set E is a τ-*cycle* with order k if there exists a positive integer k and an integer $i \in E$ such that: (i) k is the smallest positive integer such that $\pi^k(i) = i$ and (ii) π fixes each $j \in E \setminus \{i, \pi(i), \dots, \pi^{k-1}(i)\}$. The τ-cycle with order k π is usually denoted $(i, \pi(i), \dots, \pi^{k-1}(i))$. The set $\mathrm{supp}(\pi) = \{i \in E : \pi(i) \neq i\}$ denotes the support of π and we let $\mathrm{supp}(\tau)$ denote the support of the τ-cycle, that is $\mathrm{supp}(\tau) = \{i, \pi(i), \dots, \pi^{k-1}(i)\}$.

Let $\pi = \tau_1 \cdots \tau_q$ be a permutation of the set E, where τ_i, $i = 1,2,\dots,q$ are disjoint cycles and we denote by $\mathrm{ord}(\tau_i)$ the order of a cycle τ_i.

Theorem 1. *For the SCSO W (3) the following statements are hold:*

1. *The sets* $M_{\tau_i}=\left\{\mathbf{x}\in S^{m-1}:\prod_{k\in\operatorname{supp}(\tau_i)}x_k=0\right\}$, *where* $i=1,2,...,q$ *and* $M_\eta^{(ij)}=\left\{\mathbf{x}\in S^{m-1}:\prod_{k\in\operatorname{supp}(\tau_i)}x_k=\eta\prod_{k\in\operatorname{supp}(\tau_j)}x_k\vee\prod_{k\in\operatorname{supp}(\tau_i)}x_k=\frac{1}{\eta}\prod_{k\in\operatorname{supp}(\tau_j)}x_k\right\}$ *are invariant sets under* W *for any* $\eta>0$, *when* $\operatorname{ord}(\tau_i)=\operatorname{ord}(\tau_j)$, $i\neq j$ *and* $i,j\in\{1,2,\ldots,q\}$;

2. *a) If* $a\neq1$, $\pi\neq\mathbf{Id}$ *then* $\operatorname{Fix}(W)=\hat{X}\cup\{\mathbf{e}_m\}$, *where* $\mathbf{e}_m=(0,\ldots,0,1)$,

$$\hat{X}=\{\mathbf{x}\in S^{m-1}:x_k=x_l,\ \forall\ k,l\in\operatorname{supp}(\tau_i),\ i=1,\ldots,q,\ x_m=0\};$$

b) If $a=1$, $\pi\neq\mathbf{Id}$ *then we have* $\operatorname{Fix}(W)=\hat{X}$, *where*

$$\hat{X}=\bigcup_{\alpha\in(0,1)}\{\mathbf{x}\in S^{m-1}:x_k=x_l,\ \forall\ k,l\in\operatorname{supp}(\tau_i),\ i=1,\ldots,q,\ x_m=\alpha\};$$

3. *a) If* $a\neq1$, $\pi\neq\mathbf{Id}$ *then the points of the set* $\operatorname{Per}_s(W)=\{\mathbf{x}\in S^{m-1}:x_m=0\}$ *are periodic points of the SCSO* W *with period* $s=\operatorname{lcm}(\operatorname{ord}(\tau_1),\ldots,\operatorname{ord}(\tau_q))$;

b) If $a=1$ *and* $\pi\neq\mathbf{Id}$ *then the points of the set* $\operatorname{Per}_s(W)=\bigcup_{\alpha\in(0,1)}\{\mathbf{x}\in S^{m-1}:x_m=\alpha\}$ *are periodic points of the operator* W *with period* $s=\operatorname{lcm}(\operatorname{ord}(\tau_1),\ldots,\operatorname{ord}(\tau_q))$;

4. $\operatorname{Per}_n(W)=\varnothing$ *for any* $n>s$.

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