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On Dynamics of Cubic Stochastic Operators

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Abstract:

In this paper, we consider a class of cubic stochastic operators defined on a finite-dimensional simplex. Namely, cubic stochastic operators that have the form of a product of three linear operators defined on a simplex.

Keywords: Cubic stochastic operator, simplex, fixed point, limit point.

Introduction

There are many systems which are described by nonlinear operators. A quadratic stochastic operator (QSO) is one of the simplest nonlinear cases.

Let $E=\{1,2,...,m\}$ be a finite set and the set of all probability distribution on E

$S^{m-1}=\{x=(x_1,\dots,x_m)\in\mathbb{R}^m: x_i\geq 0, \sum_{i=1}^m x_i=1\}$ (1)

be the $(m-1)$-dimensional simplex. A QSO is a mapping defined as $V:S^{m-1}\to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x})=\mathbf{x}'\in S^{m-1}$, where

$$x_k'=\sum_{i,j=1}^m P_{ij,k}x_ix_j, \quad k\in E, \quad (2)$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k}=P_{ji,k}\geq 0, \quad \sum_{k=1}^m P_{ij,k}=1 \text{ for all } i,j\in E. \quad (3)$$

The trajectory (orbit) $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$, of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V\left(\mathbf{x}^{(n)}\right) = V^{(n+1)}\left(\mathbf{x}^{(0)}\right), \qquad n = 0,1,2,\ldots \ .$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. This problem was solved completely for the Volterra QSO.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of CSO (8) is defined by $\mathbf{x}^{(n+1)} = W\left(\mathbf{x}^{(n)}\right)$, where $n = 0,1,2,\ldots$ with $\mathbf{x} = \mathbf{x}^{(0)}$. Denote by $\omega\left(\mathbf{x}^{(0)}\right)$ the set of limit points of the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$. Since $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is a compact set, it is follows that $\omega\left(\mathbf{x}^{(0)}\right) \neq \varnothing$. If $\omega\left(\mathbf{x}^{(0)}\right)$ consists of a single point, they the trajectory converges and $\omega\left(\mathbf{x}^{(0)}\right)$ is a fixed point of the operator W. A point $\mathbf{x} \in S^{m-1}$ is called a fixed of the W if $W(\mathbf{x}) = \mathbf{x}$. Denote by $\mathrm{Fix}(W)$ the set of all fixed points of the operator W, i.e.

$$\mathrm{Fix}(W) = \left\{\mathbf{x} \in S^{m-1} : W(\mathbf{x}) = \mathbf{x}\right\}.$$

Let $DW\left(\mathbf{x}^{*}\right) = \left(\partial W_i / \partial x_j\right)\left(\mathbf{x}^{*}\right)$ be a Jacobian of W at the point $\mathbf{x}^{*}$.

We consider CSO (2), (3) with additional condition

$P_{ijk,l} = a_{il} b_{jl} c_{kl}$, for all $i, j, k, l \in E$, (7)

where $a_{il}, b_{jl}, c_{kl} \in \mathbb{R}$ entries of matrices $A = \left(a_{il}\right)$, $B = \left(b_{jl}\right)$ and $C = \left(c_{kl}\right)$ such that the conditions (3) are satisfied for the coefficients (7).

Then the CSO W corresponding to the coefficients (7) has the form

$x_l' = \left(W(\mathbf{x})\right)_l = \left(A(\mathbf{x})\right)_l \left(B(\mathbf{x})\right)_l \left(C(\mathbf{x})\right)_l$ (8)

where $\left(A(\mathbf{x})\right)_l = \sum_{i=1}^{m} a_{il} x_i$, $\left(B(\mathbf{x})\right)_l = \sum_{j=1}^{m} b_{jl} x_j$ and $\left(C(\mathbf{x})\right)_l = \sum_{k=1}^{m} c_{kl} x_k$.

Definition 5. *The CSO* (12) *is called separable cubic stochastic operator (SCSO).*

Let us consider the following matrices:

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ B=\begin{pmatrix}1&1&1+\frac{a}{2}\\1-a&1&1\\1&1-a&1\end{pmatrix},\ C=\begin{pmatrix}1&1+a&1\\1&1&1+a\\1-\frac{a}{2}&1&1\end{pmatrix},\ (9)$$

where $a\in[-1,1]$.

Then corresponding SCSO $W:S^2\to S^2$ is:

$$W:\begin{cases}x_1'=x_1\left(x_1+(1-a)x_2+x_3\right)\left(x_1+x_2+\left(1-\frac{a}{2}\right)x_3\right),\\x_2'=x_2\left(x_1+x_2+(1-a)x_3\right)\left((1+a)x_1+x_2+x_3\right),\\x_3'=x_3\left(\left(1+\frac{a}{2}\right)x_1+x_2+x_3\right)\left(x_1+(1+a)x_2+x_3\right).\end{cases}\quad(10)$$

Using the equation $x_1+x_2+x_3=1$ we rewrite the operator (19) as follows

$$W:\begin{cases}x_1'=x_1\left(1-ax_2\right)\left(1-\frac{a}{2}x_3\right),\\x_2'=x_2\left(1-ax_3\right)\left(1+ax_1\right),\\x_3'=x_3\left(1+\frac{a}{2}x_1\right)\left(1+ax_2\right).\end{cases}\quad(11)$$

Evidently, that if $a=0$ the SCSO (11) is the identity map. For this in the below, we consider the case when $a\neq 0$.

Let a face of the simplex S^2 be the set $\Gamma_\alpha=\left\{\mathbf{x}\in S^2: x_i=0,\ i\notin\alpha\subset\{1,2,3\}\right\}$.

Let the set $\operatorname{int}S^2=\left\{\mathbf{x}\in S^2: x_1x_2x_3>0\right\}$ and let the set $\partial S^2=S^2\setminus\operatorname{int}S^2$ be the interior and the boundary of the simplex S^2, respectively. Let $\mathbf{e}_1=(1,0,0)$, $\mathbf{e}_2=(0,1,0)$, $\mathbf{e}_3=(0,0,1)$ be the vertexes of the two-dimensional simplex.

Let $\mathbf{x}^{(0)}\in S^2$ be the initial point. Then the trajectory of $\mathbf{x}^{(0)}$ is denote by $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$ and is defined as $W\left(\mathbf{x}^{(n)}\right)=\mathbf{x}^{(n+1)}$ where $n=0,1,2,\dots$.

Theorem 1. *For the SCSO* W (11), *the following assertions true*:

(i) *If* $a \in [-1,0)$, *then*

$$\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{e}_2 & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{\{2,3\}} \setminus \{\mathbf{e}_3\}, \\ \mathbf{e}_1 & \text{if} \quad \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}. \end{cases}$$

(ii) *If* $a \in (0,1]$, *then*

$$\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{e}_2 & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_1\}, \\ \mathbf{e}_3 & \text{if} \quad \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}. \end{cases}$$

Proof: (i) If $a \in [-1,0)$. Let $\mathbf{x}^{(0)} = \left(0, x_2^{(0)}, x_3^{(0)}\right) \in \Gamma_{\{2,3\}}$. By Theorem 1, the face $\Gamma_{\{2,3\}}$ is an invariant set and the vertexes $\mathbf{e}_2$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SCSO W (11) is true $x_1^{(n)} = 0$, $n = 0,1,2,\ldots$ at the face $\Gamma_{\{2,3\}}$. The restriction of the SCSO W (11) on the face $\Gamma_{\{2,3\}}$ has the form

$$\begin{cases} x_2' = x_2\left(x_2 + (1-a)x_3\right), \\ x_3' = x_3\left((1+a)x_2 + x_3\right). \end{cases} \quad (12)$$

From the first equation of (12), one has

$$x_2^{(n+1)} = x_2^{(n)}\left(1 - a x_3^{(n)}\right) \geq x_2^{(n)}, \quad n = 0,1,2,\ldots.$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. Moreover, from the second equation of (12), one has

$$x_3^{(n+1)} = x_3^{(n)}\left(1 + a x_2^{(n)}\right) \leq x_3^{(n)}, \quad n = 0,1,2,\ldots.$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(0, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed we get $\mathbf{x}^* = \mathbf{e}_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} = \Gamma_{\{2,3\}} \setminus \{\mathbf{e}_3\}$.

Let $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{2,3\}}$. Then, from the first equation of operator (12), one has

$$x_1^{(n+1)} = x_1^{(n)}\left(1 - \frac{a}{2} x_3^{(n)}\right)\left(1 - a x_2^{(n)}\right) \geq x_1^{(n)}, \quad n = 0,1,2,\ldots.$$

So we have that there exists the $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Moreover, from the third equation of operator (12), one has

$$x_3^{(n+1)} = x_3^{(n)}\left(1+\frac{a}{2}x_1^{(n)}\right)\left(1+ax_2^{(n)}\right) \le x_3^{(n)}, \quad n = 0,1,2,\dots .$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Thus, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_1$.

Consequently, we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_1$ for any $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{2,3\}}$.

(ii) If $a \in (0,1]$. Let $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}}$. By Theorem 1, the face $\Gamma_{\{1,2\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed points belonging to this face. The restriction of the SCSO W (20) on the face $\Gamma_{\{1,2\}}$ has the form

$$\begin{cases} x_1' = x_1\left(x_1 + (1-a)x_2\right), \\ x_2' = x_2\left((1+a)x_1 + x_2\right). \end{cases} \quad (13)$$

Not that, for the SCSO W (13) is true $x_3^{(n)} = 0,\ n = 0,1,2,\dots$ at the face $\Gamma_{\{1,2\}}$. From the first equation of (13), one has

$$x_1^{(n+1)} = x_1^{(n)}\left(1-ax_2^{(n)}\right) \le x_1^{(n)}, \quad n = 0,1,2,\dots .$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Moreover, from the second equation of (13), one has

$$x_2^{(n+1)} = x_2^{(n)}\left(1+ax_1^{(n)}\right) \ge x_2^{(n)}, \quad n = 0,1,2,\dots .$$

So we have that there exists the $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. Hence there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, 0\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} = \Gamma_{\{1,2\}} \setminus \left\{\mathbf{e}_1\right\}$.

Let $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{1,2\}}$. Then, from the first equation of operator (11), one has

$$x_1^{(n+1)} = x_1^{(n)}\left(1-\frac{a}{2}x_3^{(n)}\right)\left(1-ax_2^{(n)}\right) \le x_1^{(n)}, \quad n=0,1,2,\dots .$$

Therefore, it follows that there exists the $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Moreover, from the third equation of operator (13), one has

$$x_3^{(n+1)} = x_3^{(n)}\left(1+\frac{a}{2}x_1^{(n)}\right)\left(1+ax_2^{(n)}\right) \ge x_3^{(n)}, \quad n=0,1,2,\dots .$$

Hence one has that there exists the $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Thus, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n\to\infty} x_2^{(n)} = x_2^*$. So there is the limit $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^* = \left(x_1^*, x_2^*, x_3^*\right)$. Since $\mathbf{x}^*$ should be a fixed point we get $\mathbf{x}^* = \mathbf{e}_3$.

Consequently, we have that $\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} = S^2 \setminus \Gamma_{\{1,2\}}$.

The theorem is proved. $\square$

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