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Solving Logarithmic Inequalities With the Rationalization Method

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Abstract

An inequality containing a variable only under the logarithm sign is called logarithmic.

Keywords: *logarithmic inequality, solution, least integer solution, equivalent inequalities, system of inequalities.*

Introduction

For example, inequalities of the form $\log_a f(x) > \log_a g(x),\ \log_a f(x) < \log_a g(x)$ at $a > 0,\ a \neq 1$ are logarithmic.

When solving logarithmic inequalities we will use the following properties:

1. For all admissible values a, b, c such that $a > 0,\ a \neq 1,\ b > 0,\ c > 0$ the following statements are true:

1) . inequalities $\log_a b > \log_a c \Leftrightarrow (a-1)(b-c) > 0$

2) . inequalities $\log_a b \geq \log_a c \Leftrightarrow (a-1)(b-c) \geq 0$

3) . inequalities $\log_a b < \log_a c \Leftrightarrow (a-1)(b-c) < 0$

4) . inequalities $\log_a b \leq \log_a c \Leftrightarrow (a-1)(b-c) \leq 0$

2.

1) . inequalities $\log_a b \cdot \log_c d > 0 \Leftrightarrow (a-1)(b-1)(c-1)(d-1) > 0$

2) . inequalities $\log_a b \cdot \log_c d \geq 0 \Leftrightarrow (a-1)(b-1)(c-1)(d-1) \geq 0$

3) . inequalities $\log_a b \cdot \log_c d < 0 \Leftrightarrow (a-1)(b-1)(c-1)(d-1) < 0$

4) . inequalities $\log_a b \cdot \log_c d \leq 0 \Leftrightarrow (a-1)(b-1)(c-1)(d-1) \leq 0$

3.

1) . inequalities $\log_a b-\log_c b>0 \Leftrightarrow (a-1)(b-1)(c-1)(c-a)>0$

2) . inequalities $\log_a b-\log_c b\geq 0 \Leftrightarrow (a-1)(b-1)(c-1)(c-a)\geq 0$

3) . inequalities $\log_a b-\log_c b<0 \Leftrightarrow (a-1)(b-1)(c-1)(c-a)<0$

4) . inequalities $\log_a b-\log_c b\leq 0 \Leftrightarrow (a-1)(b-1)(c-1)(c-a)\leq 0$

Problem 1. Solve the inequality

$$\log_{3x}\left(2x^2-5x+8\right)\leq \log_{3x}\left(x^2+x\right)$$

Solution: This inequality, taking into account the domain of the logarithmic function, is equivalent to the system

$$\log_{3x}\left(2x^2-5x+8\right)\leq \log_{3x}\left(x^2+x\right)\Leftrightarrow \begin{cases}(3x-1)\left(2x^2-5x+8-x^2-x\right)\leq 0\\ 3x\neq 1\\ 3x>0\\ 2x^2-5x+8>0\\ x^2+x>0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow \begin{cases}3\left(x-\frac{1}{3}\right)\left(x^2-6x+8\right)\leq 0\\ x\neq \frac{1}{3}\\ x>0\\ 2x^2-5x+8>0\\ x(x+1)>0\end{cases}\Leftrightarrow \begin{cases}\left(x-\frac{1}{3}\right)(x-2)(x-4)\leq 0\\ x\neq \frac{1}{3}\\ x>0\\ x\in R\end{cases}\Leftrightarrow$$

$$\Leftrightarrow \begin{cases}\left[\begin{array}{l}x<\frac{1}{3}\\ 2\leq x\leq 4\end{array}\right.\\ x>0\end{cases}\Leftrightarrow \left[\begin{array}{l}\begin{cases}x<\frac{1}{3}\\ x>0\end{cases}\\ \begin{cases}2\leq x\leq 4\\ x>0\end{cases}\end{array}\right.\Leftrightarrow \left[\begin{array}{l}0<x<\frac{1}{3}\\ 2\leq x\leq 4\end{array}\right.$$

Answer. $x\in\left(0;\frac{1}{3}\right)\cup[2;4].$

Problem 2. Solve the inequality

$$\frac{\log_x(x-5)-\log_x(13-x)}{\log_{x-3}x}<0$$

Solution.

To solve this inequality, instead of the numerator of the fraction, we write the product $(x-1)(x-5-(13-x))$,

and together with the denominator of the fraction - $(x-3-1)(x-1)$.

Then we obtain a system of inequalities

$$\frac{\log_x(x-5)-\log_x(13-x)}{\log_{x-3}x}<0 \Leftrightarrow \frac{\log_x(x-5)-\log_x(13-x)}{\log_{x-3}x-\log_{x-3}1}<0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{(x-1)(x-5-13+x)}{(x-3-1)(x-1)}<0 \\ x>0 \\ x-5>0 \\ x-3>0 \\ 13-x>0 \end{cases} \Leftrightarrow \begin{cases} \frac{(x-1)(2x-18)}{(x-4)(x-1)}<0 \\ x>0 \\ x>5 \\ x>3 \\ x<13 \end{cases} \Leftrightarrow \begin{cases} \frac{2(x-9)}{x-4}<0 \\ x>5 \\ x<13 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 4<x<9 \\ 5<x<13 \end{cases} \Leftrightarrow 5<x<9.$$

Answer. $x \in (5;9)$.

Problem 3. Solve the inequality

$$\log_{\frac{1}{x}}\left(\frac{x}{x+2}\right)\log_{x-3}\left(x^2+2\right) \le 0$$

Solution.

Applying the above solution method, we obtain a system of inequalities:

$$\log_{\frac{1}{x}}\frac{x}{x+2}\cdot\log_{x-3}\left(x^2+2\right)\le 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\log_{\frac{1}{x}}\frac{x}{x+2}-\log_{\frac{1}{x}}1\right)\cdot\left(\log_{x-3}\left(x^2+2\right)-\log_{x-3}1\right)\le 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left(\frac{1}{x}-1\right)\left(\frac{x}{x+2}-1\right)(x-3-1)\left(x^2+2-1\right)\le 0 \\ \frac{1}{x}>0 \\ x\ne 1 \\ \frac{x}{x+2}>0 \\ x-3>0 \\ x-3\ne 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dfrac{1-x}{x}\cdot\dfrac{-2}{x+2}\cdot(x-4)(x^2+1)\le 0 \\ x>0 \\ x\ne 1 \\ x>-2 \\ x>3 \\ x\ne 4 \end{cases} \Leftrightarrow \begin{cases} \dfrac{(x-1)(x-4)}{x(x+2)}\le 0 \\ x>3 \\ x\ne 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} -2<x<0 \\ 1\le x<4 \end{array}\right. \\ x>3 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} -2<x<0 \\ x>3 \end{cases} \\ \begin{cases} 1\le x<4 \\ x>3 \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x\in\varnothing \\ 3<x<4 \end{array}\right. \Leftrightarrow 3<x<4.$$

Answer. $x\in(3;\ 4)$.

Problem 4. Solve the inequality

$$\frac{1}{\log_{\frac{1}{6}}(3x^2-2)} > \frac{1}{\log_{\frac{1}{3}} x} + \frac{1}{\log_{\frac{1}{2}} x}$$

Solution.

Let us write this inequality in the form:

$$\frac{1}{\log_{\frac{1}{6}}(3x^2-2)} > \frac{1}{\log_{\frac{1}{3}} x} + \frac{1}{\log_{\frac{1}{2}} x} \Leftrightarrow \frac{\log_{(3x^2-2)}\frac{1}{6}}{\log_{(3x^2-2)}(3x^2-2)} > \frac{\log_x \frac{1}{3}}{\log_x x} + \frac{\log_x \frac{1}{2}}{\log_x x} \Leftrightarrow$$

$$\Leftrightarrow \log_{(3x^2-2)}\frac{1}{6} > \log_x\frac{1}{3} + \log_x\frac{1}{2} \Leftrightarrow \log_{(3x^2-2)}\frac{1}{6} > \log_x\frac{1}{6} \Leftrightarrow$$

$$\Leftrightarrow \log_{(3x^2-2)}\frac{1}{6} - \log_x\frac{1}{6} > 0 \qquad (1)$$

Let's use the rationalization method.

Inequality (1) is equivalent to the system of inequalities:

$$\Leftrightarrow \begin{cases} (3x^2-2-1)\left(\dfrac{1}{6}-1\right)(x-1)(3x^2-2-x)>0 \\ 3\left(x^2-\dfrac{2}{3}\right)>0 \\ x>0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3\left(x^2-1\right)\left(-\frac{5}{6}\right)(x-1)\left(-3x^2+x+2\right)>0 \\ x^2-\frac{2}{3}>0 \\ x>0 \end{cases} \Leftrightarrow \begin{cases} \left(x^2-1\right)(x-1)\left(3x^2-x-2\right)>0 \\ \left(x+\frac{\sqrt{6}}{3}\right)\left(x-\frac{\sqrt{6}}{3}\right)>0 \\ x>0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x-1)^3(x+1)\left(x+\frac{2}{3}\right)>0 \\ \left(x+\frac{\sqrt{6}}{3}\right)\left(x-\frac{\sqrt{6}}{3}\right)>0 \\ x>0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} -1<x<-\frac{2}{3} \\ x>1 \end{array}\right. \\ \left[\begin{array}{l} x<-\frac{\sqrt{6}}{3} \\ x>\frac{\sqrt{6}}{3} \end{array}\right. \\ x>0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} \begin{cases} -1<x<-\frac{2}{3} \\ x<-\frac{\sqrt{6}}{3} \end{cases} \\ \begin{cases} -1<x<-\frac{2}{3} \\ x>\frac{\sqrt{6}}{3} \end{cases} \\ \begin{cases} x>1 \\ x<-\frac{\sqrt{6}}{3} \end{cases} \\ \begin{cases} x>1 \\ x>\frac{\sqrt{6}}{3} \end{cases} \end{array}\right. \\ x>0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} -1<x<-\frac{\sqrt{6}}{3} \\ x\in\varnothing \\ x\in\varnothing \\ x>1 \end{array}\right. \\ x>0 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} -1<x<-\frac{\sqrt{6}}{3} \\ x>0 \end{cases} \\ \begin{cases} x>1 \\ x>0 \end{cases} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x\in\varnothing \\ x>1 \end{array}\right. \Leftrightarrow x>1.$$

Answer. $x\in(1;+\infty)$.

Problem 5. Solve the inequality

$$\log_{9-x}\left(x^2-6x+8\right)-\log_{9-x}(3x-12)>0$$

Solution.

This inequality is equivalent to a system of inequalities

$$\begin{cases}(9-x-1)(x^2-6x+8-3x+12)>0\\9-x>0\\x^2-6x+8>0\\3x-12>0\end{cases}\Leftrightarrow\begin{cases}(8-x)(x^2-9x+20)>0\\x<9\\(x-2)(x-4)>0\\x>4\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}(x-8)(x-4)(x-5)>0\\x<9\\x>4\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}x<4\\5<x<8\end{array}\right.\\x>4\end{cases}\Leftrightarrow\left[\begin{array}{l}\begin{cases}x<4\\x>4\end{cases}\\\begin{cases}5<x<8\\x>4\end{cases}\end{array}\right.\Leftrightarrow\left[\begin{array}{l}x\in\varnothing\\5<x<8\end{array}\right.\Leftrightarrow$$

$$\Leftrightarrow 5<x<8.$$

Answer. $x\in(5;\,8).$

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