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About a Minimax Control Problem for Differential Inclusion with Uncertainty Parameter

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Abstract:

In this paper the one class of controllable differential inclusions with uncertainty parameter is considered. For this model of dynamic control systems under conditions of indeterminacy the minimax control problem is researched. In this problem the necessary and sufficient conditions of optimality is obtained.

Keywords: *differential inclusion, control system, uncertainty parameter, minimax problem, optimal control, conditions of optimality.*

1. Introduction

Deterministic models do not meet the modern requirements for effective management of complex technological and economic processes, for which errors and inaccuracies of the determining process parameters are significant. To describe the dynamics of processes without the help of probabilistic characteristics of the model's uncertainties, one can use differential equations with a multi-valued right-hand side (they are called differential inclusions), i.e. relations of the form

$$\dot{x} \in F(t, x), \quad (1)$$

where x is a phase vector from the Euclidean n-dimensional space R^n, $\dot{x} = \frac{dx}{dt}$ is a velocity vector, t is time, $F(t, x)$ is a given multi-valued map, i.e., a function that corresponds to each moment of time t and each point $x \in R^n$ to the set $F(t, x)$ of R^n.

The first studies on differential equations with a multi-valued right-hand side were carried out in the works of S. Zaremba and A. Marsho in the 30 s of the XX century. Only after A. F. Filippov [1] made the first and successful application of differential equations with a multi-valued right-hand side to the problems of optimal control, since the 60s of the XX century, there was a great interest in differential inclusions. Studies of differential equations with a discontinuous right-hand side [2] have shown that the generalization of the concept of solving such equations is possible using the differential inclusion model. Differential inclusions have found applications in differential games and mathematical economics.

The questions concerning the theory of differential inclusions are very diverse. Since differential inclusions are a generalization of differential equations $\dot{x} = f(t,x)$ to the case where the right-hand side of $f(t,x)$ is multi-valueds, all the usual problems inherent in differential equations arise. These include the existence and boundedness of the solution, the continuous dependence of solutions on initial conditions and parameters, periodic solutions, etc. On the other hand, the differential inclusion has an entire family of trajectories from each starting initial point x_0. As a result, other specific questions also arise, such as the closeness, compactness, convexity and connectivity of the solution family, the properties of the integral funnel and the reachability set, and many others [3].

The theory of differential inclusions develops in various directions. We study differential-functional and integro-differential inclusions [4,5], differential inclusions with delays [6], partial differential inclusions [7], differential inclusions with a fuzzy right-hand side [8], and other classes of differential inclusions [9 –14].

2. Object of research and methods.

In the development of the theory of differential inclusions, a qualitative shift was the beginning of research on controlled differential inclusions of the form [10,12,13]

$$\frac{dx}{dt} \in F(t,x,u), u \in U \text{ , (2)}$$

where $F(t,x,u)$ is a multi-valued mapping that depends on the control parameter $u = u(t)$. Differential inclusions of the form (2) arise in the study of control systems under conditions of uncertainty. In fact, let the following control object model be given:

$$\frac{dx}{dt} = f(t,x,u,w),\ u \in U \text{ , (3)}$$

where $u = u(t)$ is the control parameter, and $w = w(t)$ is the external influence parameter, and the information about this parameter is minimal, i.e. only the area of its change is known: $w \in W$. Under fairly general assumptions, i.e. when functions $f(t,x,u,w)$ are measurable by t, continuous by (x,u,w), and $w = w(t)$ are measurable, the control system (3) is equivalent to a controlled differential inclusion

$$\frac{dx}{dt} \in f(t,x,u,W).$$

The problems for differential inclusions are of interest from the point of view of their practical application to the problems of various classes of control systems. For control systems in conditions of uncertainty, the tasks of controlling an ensemble of trajectories are of great importance. They lead to similar problems for differential inclusions to control parameters. The problems of controlling an ensemble of trajectories for control systems under conditions of uncertainty and differential inclusions are considered in [10, 12 –18]. Questions of analysis and synthesis of control systems lead to the need to study models taking into account the influence of various internal and external parameters. The class of such models includes differential inclusions with control parameters and uncertainty [17 – 20].

Consider the model of the control system in the form of the following differential inclusion

$$\dot{x} \in A(t)x + B(t,u) + D(t,q)\,,\; x(t_0) \in X_0, u \in V, q \in Q, t \in T = [t_0, t_1], \quad (4)$$

where x – n-dimensional state vector, u - m-dimensional control vector, q - k-dimensional vector of external influences, $A(t)$ - $n \times n$-matrix, $B(t,u) \subset R^n$, $D(t,q) \subset R^n$, $X_0 \subset R^n$, $V \subset R^m$, $Q \subset R^k$. With respect to parameter q, we will assume that it is constant over the interval $T = [t_0, t_1]$, but only its set of possible values Q is known .

The control system (4) will be studied under the following assumptions:

1) the elements of matrix $A(t)$ are summable by $T = [t_0, t_1]$; 2) $X_0 \subset R^n$, $V \subset R^m$, $Q \subset R^k$ – convex compacts; 3) for any $t \in T = [t_0, t_1]$, $u \in V, q \in Q$ sets $B(t,u)$ and $D(t,q)$ non-empty compacts of R^n ; 4) the multi-valued map $(t,u) \to B(t,u)$ is measurable by $t \in T = [t_0, t_1]$, continuous by $u \in V$, and integrally bounded, i.e. there is a function $T = [t_0, t_1]$ summable by $\beta_B(t)$ such that $\sup\{\|\gamma\| : \gamma \in B(t,u)\} \le \beta_B(t)\}, (t,u) \in T \times V$; 5) the multi-valued map $(t,q) \to D(t,q)$ is measurable by $t \in T = [t_0, t_1]$, continuous by $q \in Q$, and integrally bounded, i.e. there is a function $T = [t_0, t_1]$ summable by $\beta_D(t)$ such that $\sup\{\|\gamma\| : \gamma \in B(t,u)\} \le \beta_D(t)$, $(t,u) \in T \times V$.

Definition 1. By admissible controls for system (4), we mean measurable bounded m-vector functions $u = u(t)$, $t \in T = [t_0, t_1]$, which take almost everywhere by $T = [t_0, t_1]$ values from the convex compact V .

Definition 2. An admissible trajectory corresponding to control $u = u(t)$, $t \in T = [t_0, t_1]$, and parameter $q \in Q$ is an absolutely continuous n-vector function $x(t) = x(t,u,q)$ that satisfies almost everywhere on $T = [t_0, t_1]$ the differential inclusion (4) and the initial condition $x(t_0) \in X_0$.

Let: U – the set of all permissible controls; $H(u,q)$ – the set of admissibly trajectories corresponding to the pair $(u,q) \in U \times Q$; $X(t,u,q) = \{\xi \in R^n : \xi = x(t),\; x(\cdot) \in H(u,q)\}$. For model (4), topological properties of sets $H(u,q)$ and $X(t,u,q)$ are important . Using the methods of [20] , we can study the convexity and compactness properties of sets $H(u,q)$ and $X(t,u,q)$, as well as the continuity, closure, and convexity of multi-valued maps: $(u,q) \to H_T(u,q)$, $(t,u,q) \to X(t,u,q)$.

Consider the set $X_1(u,q) = X(t_1,u,q)$, which is the set of all such points of the state space that can be reached at moment t_1, moving along the admissible trajectories of $x(\cdot) \in H(u,q)$. We will manage this terminal state $X_1(u,q)$ by evaluating the quality of the control process using the following non-smooth functionality:

$$g(X_1(u,q)) = \sup\{\sum_{i=1}^{l} \min_{z_i \in Z_i}(z_i, P\xi) : \xi \in X_1(u,q)\}, \quad (5)$$

where $P - s\times n$ is a matrix, Z_i, $i=\overline{1,l}$, are compacts of R^s. Given that in the system (4), parameter $q\in Q$ has the character of uncertainty, the terminal state $X_1(u,q)$ will be controlled according to the minimax principle, i.e., according to the principle of obtaining a guaranteed result. So, the following minimax problem is considered

$$\sup_{q\in Q} g(X_1(u,q)) \to \min,\ u\in U. \qquad (6)$$

Control $u^*\in U$, minimizing non-smooth functionality of the form

$$J(u)=\sup_{q\in Q} g(X_1(u,q)) \quad (7)$$

we call optimal control in problem (6). Here we will study the necessary and sufficient conditions optimality in nonsmooth control problem (6).

Let $F(t,\tau)$ be the fundamental matrix of solutions to equation $\dot{x}=A(t)x$, $F(\tau,\tau)=E$. According to the results of the theory of differential inclusions, for any $(u,q)\in U\times Q$, the set $X_1(u,q)$ has the representation:

$$X_1(u,q)=F(t_1,t_0)X_0+\int_{t_0}^{t_1} F(t_1,t)[B(t,u(t))+D(t,q)]dt. \qquad (8)$$

Let $C(Y,\psi)=\sup_{y\in Y}(y,\psi)$ be a support function of a bounded set $Y\subset R^n$. Using formula (8), the properties of the support functions and the integral of multi-valued maps, we obtain that for any $(u,q)\in U\times Q$, the set $X_1(u,q)$ is a convex compact of R^n and its support function has the form:

$$C(X_1(u,q),\psi)=C(F(t_1,t_0)X_0,\psi)+\int_{t_0}^{t_1} C(F(t_1,t)[B(t,u(t))+D(t,q)],\psi)dt. \qquad (9)$$

Further, since

$$\sum_{i=1}^{l}\min_{z_i\in Z_i}(z_i,P\xi)=\min_{z_i\in Z_i,\,i=\overline{1,l}}\Big(P\xi,\sum_{i=1}^{l} z_i\Big)=\min_{z\in coZ}(P\xi,z)=\min_{z\in coZ}(\xi,P'z),$$

then, using the formula (9) and the minimax theorem from convex analysis, we obtain that for the functional (5) the formula is valid

$$g(X_1(u,q))=\min_{z\in coZ}\{C(F(t_1,t_0)X_0,P'z)+\int_{t_0}^{t_1} C(F(t_1,t)[B(t,u(t))+D(t,q)],P'z)dt\}, \quad (10)$$

где $z=\sum_{i=1}^{l} z_i$, $Z=\sum_{i=1}^{l} Z_i$, coZ –the convex hull of the set Z.

3. Main results.

In the future, we will assume that, along with the conditions 1) – 5), the condition is satisfied: 6) the support function of the set $D(t,q)$ is concave by $q\in Q$.

Using (7), (9), and (10), we have:

$$J(u)=\max_{q\in Q}\min_{z\in coZ}\gamma(u,q,z), \qquad (11)$$

where

$$\gamma(u,q,z) = C(X_0, \psi(t_0,z)) + \int_{t_0}^{t_1} C(B(t,u(t)) + D(t,q), \psi(t,z))dt,$$

$\psi(t,z)$ – absolutely continuous solution of system $\dot{\psi} = A'(t)\psi,\ \psi(t_1) = P'z$. Functional $\gamma(u,q,z)$ is concave by $q \in Q$ and convex by $z \in coZ$. Therefore, applying the minimax theorem mentioned above, we obtain the following formula from (11):

$$J(u) = \min_{z \in coZ}[C(X_0, \psi(t_0,z)) + \max_{q \in Q} \int_{t_0}^{t_1} C(B(t,u(t)) + D(t,q), \psi(t,z))dt]. \qquad (12)$$

Let's introduce the following functionals:

$$\mu(u,z) = C(X_0, \psi(t_0,z)) + \int_{t_0}^{t_1} C(B(t,u(t)), \psi(t,z))dt + \max_{q \in Q} \int_{t_0}^{t_1} C(D(t,q), \psi(t,z))dt, \qquad (13)$$

$$\rho(q,z) = C(X_0, \psi(t_0,z)) + \int_{t_0}^{t_1} \min_{v \in V} C(B(t,v), \psi(t,z))dt + \int_{t_0}^{t_1} C(D(t,q), \psi(t,z))dt. \qquad (14)$$

Theorem 1. Let: 1) $u^*(t), t \in T = [t_0, t_1]$, – optimal control in the problem (6);

2) $z^* \in coZ$ – point of the global minimum of the function $z \to \mu(u^*, z)$.

Then for almost all $t \in T = [t_0, t_1]$ the equality holds

$$\min_{v \in V} C(B(t,v), \psi(t,z^*)) = C(B(t,u^*(t)), \psi(t,z^*)). \qquad (15)$$

Proof. Since $u^*(t), t \in T = [t_0, t_1]$, is the optimal control in problem (6), then

$J(u^*) \le J(u),\ \forall u \in U$, where $J(u)$ has the form (7). Therefore, using (12) and (13), we have:

$$\min_{z \in coZ} \mu(u^*, z) \le \min_{z \in coZ} \mu(u,z),\ \forall u \in U. \qquad (16)$$

Let $z^* \in coZ$ be an point of the global minimum of function $z \to \mu(u^*, z)$. Then from (16) we get

$$\int_{t_0}^{t_1} C(B(t,u^*(t)), \psi(t,z^*))dt \le \int_{t_0}^{t_1} C(B(t,u(t)), \psi(t,z^*))dt,\ \forall u \in U.$$

Therefore,

$$\int_{t_0}^{t_1} C(B(t,u^*(t)), \psi(t,z^*))dt \le \int_{t_0}^{t_1} \min_{v \in V} C(B(t,v), \psi(t,z^*))dt.$$

From here we get

$$\int_{t_0}^{t_1} C(B(t,u^*(t)), \psi(t,z^*))dt = \int_{t_0}^{t_1} \min_{v \in V} C(B(t,v), \psi(t,z^*))dt. \qquad (17)$$

Due to the properties of the Lebesgue integral, it follows from (17) that equality (15) holds for almost all $t \in T = [t_0, t_1]$.

Let us use the result of the proved theorem. First, given (7) and (12), we have a chain of equalities:

$$\min_{z\in coZ}[C(X_0,\psi(t_0,z))+\int_{t_0}^{t_1}\min_{v\in V}C(B(t,v),\psi(t,z))dt+\max_{q\in Q}\int_{t_0}^{t_1}C(D(t,q),\psi(t,z))dt]=$$

$$=\min_{u\in U}\min_{z\in coZ}[C(X_0,\psi(t_0,z))+\int_{t_0}^{t_1}C(B(t,u(t)),\psi(t,z))dt+\max_{q\in Q}\int_{t_0}^{t_1}C(D(t,q),\psi(t,z))dt]=$$

$$=\min_{u\in U}\max_{q\in Q}g(X_1(u,q)). \qquad (18)$$

Next, let $u^*(t), t\in T=[t_0,t_1]$ be the optimal control in problem (6), and $z^*\in coZ$ be the point of the global minimum of function $z\to\mu(u^*,z)$. Then, using (7), (12), (13), (15), we have:

$$\min_{u\in U}\max_{q\in Q}g(X_1(u,q))=\min_{u\in U}\min_{z\in coZ}\mu(u,z)=\min_{z\in coZ}\mu(u^*,z)=\mu(u^*,z^*)=$$

$$=C(X_0,\psi(t_0,z^*))+\int_{t_0}^{t_1}C(B(t,u^*(t)),\psi(t,z^*))dt+\max_{q\in Q}\int_{t_0}^{t_1}C(D(t,q),\psi(t,z^*))dt=$$

$$=C(X_0,\psi(t_0,z^*))+\int_{t_0}^{t_1}\min_{v\in V}C(B(t,v),\psi(t,z^*))dt+\max_{q\in Q}\int_{t_0}^{t_1}C(D(t,q),\psi(t,z^*))dt\geq$$

$$\geq\min_{z\in coZ}[C(X_0,\psi(t_0,z))+\int_{t_0}^{t_1}\min_{v\in V}C(B(t,v),\psi(t,z))dt+\max_{q\in Q}\int_{t_0}^{t_1}C(D(t,q),\psi(t,z))dt. \qquad (19)$$

Now, taking into account the definition (14) of the functional $\rho(q,z)$, from (18) and (19) we get that z^* is the point of the global minimum of the function $z\to\max_{q\in Q}\rho(q,z)$. Therefore, the following necessary optimality conditions are valid.

Theorem 2. Let $u^*(t), t\in T=[t_0,t_1]$, be the optimal control in problem (6). Then there exists a $z^*\in coZ$ – point of the global minimum of the function $z\to\max_{q\in Q}\rho(q,z)$ by coZ , such that, for almost all $t\in T=[t_0,t_1]$, equality (15) holds.

Now we give sufficient optimality conditions in problem (6).

Theorem 3. Let $z^*\in coZ$ be the point of the global minimum of a function $z\to\max_{q\in Q}\rho(q,z)$ by coZ, and $u^*(t), t\in T=[t_0,t_1]$ be a admissible control satisfying almost everywhere on $T=[t_0,t_1]$ the relation (15). Then $u^*(t)$ is the optimal control in problem (6).

4. Discussion of the results and conclusion.

The necessary optimality condition for the considered non-smooth minimax-type problem is given in Theorem 1 in the form of relation (15). It should be noted that in the case where $(t,u)\to B(t,u)$ is a single-valued mapping, condition (15) takes the form well known from the Pontryagin maximum principle [3].

To apply condition (15), the global minimum point $z^*\in coZ$ of function $z\to\mu(u^*,z)$ must be known . And in theorem 2, it is stated that the point $z^*\in coZ$ used in condition (15) is the point of the global minimum of function $z\to\max_{q\in Q}\rho(q,z)$. Since minimizing function $z\to\max_{q\in Q}\rho(q,z)$

does not require any information about the desired optimal control, theorem 2 can be considered more convenient from the point of view of practical application.

Theorem 3 actually states that the optimality conditions obtained in Theorem 2 are sufficient. So, combining theorems 2 and 3, we can say that the main result of the work is the following **optimality criterion:** in order for the admissible control $u^*(t), t \in T$ to be optimal in problem (6), it is necessary and sufficient that the point $z^* \in coZ$ of the global minimum of function $z \to \max_{q \in Q} \rho(q, z)$ exists and that the relation (16) is performed almost everywhere on $T = [t_0, t_1]$.

In conclusion, we note that in the work developing the research methods [18], the results are obtained, which are the theoretical basis for the development of an algorithm for constructing optimal control in the considered minimax problem.

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