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Anti-Symmetric Vibrations of a Free-Supported Three-Layer Elastic Plate

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Abstract:

In the study, the head parts of the slopes of the selected three-tiered elastic plate are selected as search functions. Thus, a system of fifth order differential equations, which can be used to solve practical problems, has been created and mathematical operations are performed, and this system of equations can be eliminated by the Maple 12 program, which can occur in three-dimensional plate positions and tension graphs.

Key Words: boundary conditions, three-layer plate, displacement,wave equations

Introduction

Currently, numerous research works are being conducted on multilayer structures, particularly three-layer plates. This is due to the high stiffness of three-layer plates during various vibration processes and the ease of solving economic problems. Many articles contribute to this body of work, including [1,2].

Multilayer structures, especially three-layer plates, are widely used in various fields of technology and construction. In many cases, the dynamic analysis of plates is conducted based on classical theories relying on Kirchhoff's hypotheses [3]. In some instances, dynamic calculations are based on refined S.P. Timoshenko-type equations that take into account transverse shear deformation and rotational inertia [4].

Over the last few decades, plate theories based on the exact solution method by G.I. Petrashen have been developed. In particular, using this method, symmetric structure theories for three-layer plates have been created by Professor I.G. Filippov and his students.

In this paper, the vibration equations of a three-layer elastic plate are derived using the aforementioned Petrashen–Filippov method, but considering the problem as a plane issue. Along with the vibration equations, an algorithm is developed that allows for the determination of stress deformation states at any cross-section of the plate with respect to coordinates and time.

Problem Statement

We consider a three-layer plate in the Cartesian coordinate system. The layers of the plate consist of different materials, and the contact between them is assumed to be perfect. The plate is analyzed in a state of planar deformation, viewed in its rectangular coordinates (Figure 1). The axes are directed along the contact line of the layers in the cross-section, and the vertical axis is directed perpendicular to it. The layers are numbered as "1," "2," and "3." Let the thicknesses of the layers be denoted as h_0, h_1 va h_2, with the Lame coefficients for the materials of the layers being (λ_0, μ_0), (λ_1, μ_1) , and (λ_2, μ_2), and their densities as ρ_0, ρ_1 va ρ_2.

The relationships between stresses and deformations at the points in the layers, as well as the equations of motion for layer points in the Cartesian coordinate system, are as follows:

$$\begin{aligned} \sigma_{ii}^{(m)} &= \lambda_m \left(\varepsilon^{(m)}\right) + 2\mu_m \left(\varepsilon_{ii}^{(m)}\right), \\ \sigma_{ij}^{(m)} &= \mu_m \left(\varepsilon_{ij}^{(m)}\right), \end{aligned} \tag{1}$$

$$\sigma_{ij,j}^{m} + \rho_m \cdot F_i^m = \rho_m \cdot \frac{\partial^2 U_{mi}}{\partial t^2} \quad (i, j = x, y, z) \tag{2}$$

here, the indices $m = 0,1,2$ - denote the layer numbers.

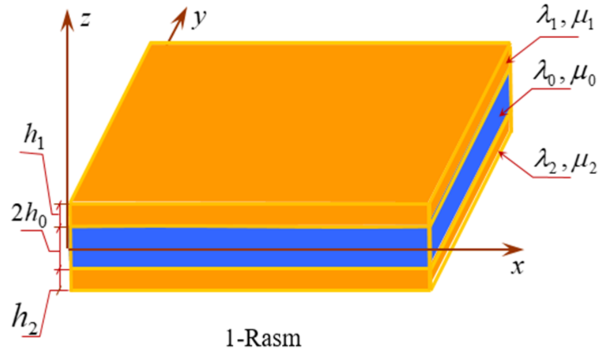
Considering that the potentials of transverse and longitudinal waves[6] are the displacement vectors $\vec{U}^m = \vec{U}^m(U_m, W_m)$ of layer points in the state of planar deformation, we introduce[7]:

$$\vec{U}^m = grad\,\varphi_m + rot\,\vec{\psi}_m \tag{3}$$

Here, $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors of the coordinate axes. Substituting these expressions (3) into the equations of motion (2) leads us to the wave equations.

$$\Delta\varphi_m = \frac{1}{a_m^2}\frac{\partial^2 \varphi_m}{\partial t^2}; \qquad \Delta\psi_m = \frac{1}{b_m^2}\frac{\partial^2 \psi_m}{\partial t^2}, \tag{4}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ is the two-dimensional Laplace differential operator.



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We assume that the plate is in a static state at time $t < 0$, and at $t = 0$, dynamic loads begin to act on its boundary surfaces. The boundary conditions then take the following form:

When $z = \pm h_m$

$$\sigma_{xz}^m = f_x^m(x,t); \quad \sigma_{zz}^m = f_z^m(x,t); \quad \sigma_{yz}^m = 0, \quad (m = 0,1,2). \tag{5}$$

Additionally, the following kinematic conditions are valid at the contact surfaces of the layers:

$$U_0(x,z,t)\big|_{z=h_0} = U_1(x,z,t)\big|_{z=h_0}; \quad W_0(x,z,t)\big|_{z=h_0} = W_1(x,z,t)\big|_{z=h_0}. \tag{6}$$

The initial conditions are assumed to be zero, that is, at t=0.

$$\varphi_m = \psi_m = 0; \qquad \frac{\partial \varphi_m}{\partial t} = \frac{\partial \psi_m}{\partial t} = 0 \tag{7}$$

This, the problem of longitudinal vibrations of the three-layer plate is reduced to integrating the system of equations (7) under the boundary conditions (5), (6), and the initial conditions.

The solution to the problem.

To solve the problem, we need to use the potential functions for ψ_m and φ_m.[5]

$$\varphi_m = \int_0^\infty \left.\begin{matrix} \sin kx \\ -\cos kx \end{matrix}\right\} dk \int_{(l)} \tilde{\varphi}_m e^{pt} dp; \quad \psi_m = \int_0^\infty \left.\begin{matrix} \cos kx \\ \sin kx \end{matrix}\right\} dk \int_{(l)} \tilde{\psi}_m e^{pt} dp, \quad (m = 0,1,2). \tag{8}$$

we will express them in the form and substitute them into [4].

$$\frac{d^2\tilde{\varphi}_m}{dz^2} - \alpha_m^2 \tilde{\varphi}_m = 0; \quad \frac{d^2\tilde{\psi}_m}{dz^2} - \alpha_m^2 \tilde{\psi}_m = 0 \quad (m = 0,1,2) \tag{9}$$

we will derive the equations. Here:

$$\alpha_m^2 = k^2 + \frac{1}{a_m^2} p^2; \quad \beta_m^2 = k^2 + \frac{1}{b_m^2} p^2 \tag{10}$$

The above (5) shows that under symmetric loading, the plate oscillates longitudinally, and the solutions to the equations (9) are composed of

$$\tilde{\varphi}_m(z,k,p) = A_m^{(1)} ch\alpha_m z, \quad \tilde{\psi}_m(z,k,p) = B_m^{(1)} sh\beta_m z. \quad (m = 0,1,2) \tag{11}$$

The displacements of the points of the layers can also be represented in the form of (8), and we will have expressions for the altered $\tilde{U}_m$, $\tilde{W}_m$.

$$\tilde{U}_m = kA_m^{(1)} ch(\alpha_m z) - \beta_m B_m^{(1)} ch(\beta_m z); \quad \tilde{W}_m = \alpha_m A_m^{(1)} sh(\alpha_m z) - kB_m^{(1)} sh(\beta_m z). \ (m = 0,1,2) \tag{12}$$

We will expand the right-hand side of these (12) expressions in a series based on the degrees of $(\alpha_m z)$ and $(\beta_m z)$

$$\tilde{W}_m = \sum_{n=0}^{\infty} \left[\alpha_m^{2n+2} A_m^{(1)} - k\beta_m^{2n+1} B_m^{(1)}\right] \frac{z^{2n+1}}{(2n+1)!}; \quad \tilde{U}_m = \sum_{n=0}^{\infty} \left[k\alpha_m^{2n} A_m^{(1)} - \beta_m^{2n+1} B_m^{(1)}\right] \frac{z^{2n}}{(2n)!} \tag{13}$$

We will choose the boundary terms of the displacements $\tilde{U}_1$ and $\tilde{W}_1$ as the sought functions in the equations of motion for the three-layered plate, that is...

$$\tilde{U}_0^{(0)} = kA_0^{(1)} - \beta_0 B_0^{(1)}, \ \tilde{W}_0^{(0)} = \left[\alpha_0^2 A_0^{(1)} - k\beta_0 B_0^{(1)}\right]\xi.$$

From here,

$$A_0^{(1)} = \frac{\frac{1}{\xi}\tilde{W}_0^{(0)} - k\tilde{U}_0^{(0)}}{\alpha_0^2 - k^2}; \qquad \beta_0 B_0^{(1)} = \frac{\frac{k}{\xi}\tilde{W}_0^{(0)} - \alpha_0^2 \tilde{U}_0^{(0)}}{\alpha_0^2 - k^2}; \tag{14}$$

By substituting the above (12) expressions for the altered displacements $\tilde{U}_m$ and $\tilde{W}_m$ into the (6) contact conditions, we obtain a system of equations. Solving this system will allow us to express the constants $A_1^{(1)}$ and $B_1^{(1)}$ in terms of $A_0^{(1)}$ and $B_0^{(1)}$.

After this, by substituting the derived expressions into (13), we can further analyze or simplify the results accordingly.

$$A_1^{(1)} = \frac{1}{(\alpha_0^2 - k^2)\Delta_1^0}\left[\frac{1}{\xi}\left(\Delta_{11}^0 + \frac{k}{\beta_0}\Delta_{12}^0\right)\tilde{W}_0^{(0)} - \left(k\Delta_{11}^0 + \frac{\alpha_0^2}{\beta_0}\Delta_{12}^0\right)\tilde{U}_0^{(0)}\right];$$

$$B_1^{(1)} = \frac{1}{(\alpha_0^2 - k^2)\Delta_1^0}\left[\frac{1}{\xi}\left(\Delta_{21}^0 + \frac{k}{\beta_0}\Delta_{22}^0\right)\tilde{W}_0^{(0)} - \left(k\Delta_{21}^0 + \frac{\alpha_0^2}{\beta_0}\Delta_{22}^0\right)\tilde{U}_0^{(0)}\right]. \tag{15}$$

To find the non-zero stresses $\sigma_{xz}^{(m)}$, $\sigma_{zz}^{(m)}$ at an arbitrary point in the plate layers, we will express them similarly to (8). Then, we substitute (8) into (1) from the opposite side, equating it with the expression described in (8).

$$\tilde{M}_1\left(2k\alpha_1 A_1^{(1)}(k,p) sh(\alpha_1 z) - (\beta_1^2 + k^2) B_1^{(1)}(k,p) sh(\beta_1 z)\right) = \tilde{f}_x^{(1)}(k,p);$$

$$\left[\tilde{L}_1(\alpha_1^2 - k^2) + 2\tilde{M}_1 k^2\right] A_1^{(1)}(k,p) ch(\alpha_1 z) - 2\tilde{M}_1 k\beta_1 B_1^{(1)}(k,p) ch(\beta_1 z) = \tilde{f}_z^{(1)}(k,p). \tag{16}$$

By substituting the values of $A_1^{(1)}$ and $B_1^{(1)}$ determined by the formulas (15) into the relationships (16), and expanding the hyperbolic functions in the resulting equations into power series based on the levels of the thickness coordinate, we obtain the general equations for longitudinal vibrations of the three-layered plate.

Since the orders of the derivatives in these equations are infinite, we assume that the conditions for truncating infinite series, as presented in [6], are satisfied. Thus, we limit ourselves to the first terms in the expansions.

As a result, we arrive at the following system of equations that can be applied in practical problems.

$$-\left\{\left[A_{11}\frac{(h_0+h_1)h_0^4}{12}+A_{12}\frac{(h_0+h_1)^3h_0^2}{36}\right]\frac{\partial^4}{\partial t^4}-\left[A_{13}\frac{(h_0+h_1)h_0^4}{12}+A_{14}\frac{(h_0+h_1)^3h_0^2}{36}\right]\frac{\partial^4}{\partial x^2\partial t^2}+\right.$$

$$+\left[A_{15}\frac{(h_0+h_1)h_0^4}{12}+A_{16}\frac{(h_0+h_1)^3h_0^2}{36}\right]\frac{\partial^4}{\partial x^4}+\left[A_{17}\frac{(h_0+h_1)h_0^2}{6}+A_{18}\frac{(h_0+h_1)^3}{6}\right]\frac{\partial^2}{\partial t^2}-$$

$$\left.-\left[A_{19}\frac{(h_0+h_1)h_0^2}{6}+A_{110}\frac{(h_0+h_1)^3}{6}\right]\frac{\partial^2}{\partial x^2}+A_{111}(h_0+h_1)\right\}\frac{1}{\xi}\frac{\partial}{\partial x}W_0^{(0)}-=\left\{\left[B_{11}\frac{(h_0+h_1)h_0^2}{6}+B_{12}\frac{(h_0+h_1)^3}{6}\right]\frac{\partial^4}{\partial t^4}-\right.$$

$$\left[B_{13}\frac{(h_0+h_1)h_0^2}{6}+B_{14}\frac{(h_0+h_1)^3}{6}\right]\frac{\partial^4}{\partial x^2\partial t^2}+$$

$$\left.+\left[B_{15}\frac{(h_0+h_1)h_0^2}{6}-B_{16}\frac{z^3}{6}\right]\frac{\partial^4}{\partial x^4}+B_{17}(h_0+h_1)\frac{\partial^2}{\partial t^2}-B_{18}(h_0+h_1)\frac{\partial^2}{\partial x^2}\right\}U_0^{(0)}=$$

$$=\left\{S_1\frac{h_0^4}{12}\frac{\partial^4}{\partial t^4}-S_2\frac{h_0^4}{12}\frac{\partial^4}{\partial x^2\partial t^2}+\frac{h_0^4}{12}\frac{\partial^4}{\partial x^4}+S_3\frac{h_0^2}{6}\frac{\partial^2}{\partial t^2}-S_4\frac{h_0^2}{6}\frac{\partial^2}{\partial x^2}+1\right\}f_x^{(1)}(k,p);\quad(17)$$

$$\left\{\left[A_{21}\frac{h_0^4}{12}+A_{22}\frac{h_0^2(h_0+h_1)^2}{12}\right]\frac{\partial^4}{\partial t^4}-\left[A_{23}\frac{h_0^4}{12}-A_{24}\frac{h_0^2(h_0+h_1)^2}{12}\right]\frac{\partial^4}{\partial x^2\partial t^2}+\right.$$

$$+\left[A_{25}\frac{h_0^4}{12}+A_{26}\frac{h_0^2(h_0+h_1)^2}{12}\right]\frac{\partial^4}{\partial x^4}+\left[A_{27}\frac{h_0^2}{6}+A_{28}\frac{(h_0+h_1)^2}{2}\right]\frac{\partial^2}{\partial t^2}-$$

$$\left.-\left[A_{29}\frac{h_0^2}{6}+A_{210}\frac{(h_0+h_1)^2}{2}\right]\frac{\partial^2}{\partial x^2}+A_{211}\right\}\frac{1}{\xi}W_0^{(0)}+\left\{\left[B_{21}\frac{h_0^4}{12}+B_{22}\frac{h_0^2(h_0+h_1)^2}{12}\right]\frac{\partial^4}{\partial t^4}-\right.$$

$$-\left[B_{23}\frac{h_0^4}{12}+B_{24}\frac{h_0^2(h_0+h_1)^2}{12}\right]\frac{\partial^4}{\partial x^2\partial t^2}+\left[B_{25}\frac{h_0^4}{12}+B_{26}\frac{h_0^2(h_0+h_1)^2}{12}\right]\frac{\partial^4}{\partial x^4}+$$

$$\left.+\left[B_{27}\frac{h_0^2}{6}+B_{28}\frac{(h_0+h_1)^2}{2}\right]\frac{\partial^2}{\partial t^2}-\left[B_{29}\frac{h_0^2}{6}+B_{210}\frac{(h_0+h_1)^2}{2}\right]\frac{\partial^2}{\partial x^2}-B_{211}\right\}\frac{\partial}{\partial x}U_0^{(0)}=$$

$$=\left\{S_1\frac{h_0^4}{12}\frac{\partial^4}{\partial t^4}-S_2\frac{h_0^4}{12}\frac{\partial^4}{\partial x^2\partial t^2}+\frac{h_0^4}{12}\frac{\partial^4}{\partial x^4}+S_3\frac{h_0^2}{6}\frac{\partial^2}{\partial t^2}-S_4\frac{h_0^2}{6}\frac{\partial^2}{\partial x^2}+1\right\}f_z^{(1)}(k,p).$$

Here, A_{ij}, B_{ij}, S_{ij} are constants that depend on the elastic properties of the layers, as indicated in $(i, j = 1,2)$.

To solve the system of equations (17) for the vibrations of a freely supported three-layered plate, we incorporate the boundary conditions pertinent to the free-supported case. By using the $h_0 = 0.05$, $h_1 = 0.0025$, $h_2 = 0.0025$, $a_0 = 0.38$, $a_2 = 1$, $b_0 = 0.26$, $b_1 = 0.5$, $b_2 = 0.5$, $z_0 = 0.05$, $z_1 = 0.05$, $z_2 = 0.05$, $\xi = 0.015$; $f_x = 1.9\cdot10^{-10}$, $f_z = 1.9\cdot10^{-10}$ "Maple 12" software, we can solve this system and determine the sought functions.

This will allow us to find the displacements and stresses that develop within the layers of the three-layered plate.

For example, the displacements of the middle layer are expressed through the sought functions as follows.

$$\begin{aligned}U_0(x,t)&=\left[(1-q_0)\frac{z^2}{2}\frac{\partial^2}{\partial t^2}-(1-q_0)\frac{z^2}{2}\frac{\partial^2}{\partial x^2}+1\right]U_0^{(0)}(x,t)-\frac{1}{\xi}q_0\frac{z^2}{2}\frac{\partial}{\partial x}W_0^{(0)}(x,t);\\ W_0(x,t)&=q_0\frac{z^3}{6}\left[\frac{\partial^3}{\partial t^2\partial x}-\frac{\partial^3}{\partial x^3}\right]U_0^{(0)}(x,t)+\frac{1}{\xi}\left[\left(\frac{1}{b_0^2}+q_0\right)\frac{z^3}{6}\frac{\partial^2}{\partial t^2}-(1+q_0)\frac{z^3}{6}\frac{\partial^2}{\partial x^2}+z\right]W_0^{(0)}(x,t)\end{aligned} \quad (18)$$

By substituting the obtained functions for $U_0^{(0)}(x,t)$ and $W_0^{(0)}(x,t)$ into these expressions, we can obtain the three-dimensional graphs of the displacements in the middle layer. Alternatively, we can

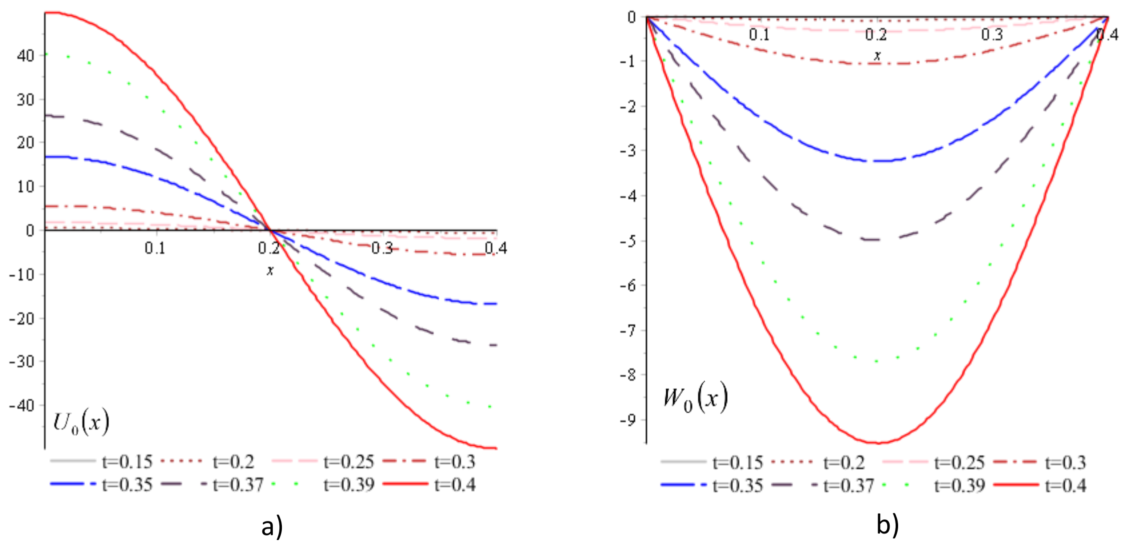

Figure 2. Graphs of the Displacements of the Middle Layer $U_0(x,t)$ and $W_0(x,t)$ Their Variation with Respect to Coordinates.

generate graphs showing how the displacements in the middle layer change with respect to the coordinates for various values of time.

Obtained Results

In this context, Figure 2 depicts the graphs of the displacements $W_0(x)$ and $U_0(x)$ of the middle layer points in a three-layered free plate, oriented along the axes. These graphs illustrate that as time increases, the displacement graphs also increase accordingly. This behavior highlights the dynamic response of the plate under vibrational conditions.

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