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# Application of the Method of Straight Lines for Solving Parabolic Equations with Arbitrary Linear Boundary Conditions

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**Abstract:**

A method for applying the straight-line technique by transforming a problem with arbitrary linear boundary conditions into a Dirichlet problem is developed. Assuming the boundary values of the desired function are given, the Dirichlet problem is solved. The actual boundary values of the desired functions are found by aligning the assumed boundary values with the newly obtained values according to boundary condition approximations. These values are then used to implement the straight-line method, ensuring second-order accuracy for equation and boundary condition approximations.

**Keywords:** Finite Difference Method, Heat Transfer, Dirichlet Problem, Eigenvalues and Vectors.

## Introduction

The numerous modifications of the finite difference method used for numerically solving single- and multi-dimensional equations of mathematical physics are considered approximate methods. On one hand, this is associated with the approximation of equations and boundary conditions to a certain degree of accuracy. On the other hand, solving finite difference equations is of an approximate nature, as methods such as discretization, variable direction, predictor-corrector, and others do not provide an exact solution but rather an approximation to it. Within the framework of solving constructed finite difference equations, the straight-line method is more efficient, as it ensures the accuracy of solutions to finite difference equations within the precision limits of machine calculations. In the works of V.N. Faddeeva [1] and S. Karimberdiyeva [2], algorithms for applying the straight-line method to solve elliptic, parabolic, and hyperbolic types of two- and three-dimensional equations under various boundary conditions have been proposed. Unfortunately, only auxiliary matrices used for solving Dirichlet problems were provided, and no information was given regarding these matrices for other boundary conditions. Auxiliary matrices—comprising fundamental and diagonal matrices—consist of eigenvalues and eigenvectors that transition from the differential operator to the finite difference operator and tri-diagonal matrix numbers. In certain works by the authors of this scientific report, auxiliary matrices for mixed boundary conditions of the first and second types have been developed. An algorithm for solving the eigenvalue and vector problem of the Dirichlet problem for the parabolic equation is presented in [3]. It has been proven that the double inequality for the eigenvalues is valid.

**Methodology**

The eigenvalues of the transition matrix to finite difference equations,

$\lambda_s = -2\left(1+\cos\frac{2s+1}{2(N+1)}\pi\right)$ The eigenvalues of the transition matrix to finite difference equations have been determined in a specific form, and the elements of the eigenvectors have been identified, where 0 and N+1 denote the numbers of the boundary nodes of the segment. We can continue the work on applying the straight-line method for other combinations of boundary conditions. Each time, new auxiliary matrices are constructed based on the developed transition matrices. The question arises: Is it possible to construct a universal algorithm for solving problems with arbitrary combinations of linear boundary conditions? Below, a positive answer to this question is provided, along with methods for approximating equations and various boundary conditions with second-order accuracy when solving problems for parabolic equations, including the Dirichlet problem using auxiliary matrices. To avoid confusion in describing the algorithm, a one-dimensional inhomogeneous parabolic equation is taken as the object of application. The main factors are explained within the framework of classical heat transfer theory. The essence of the method is as follows: Initially, the problem is solved by assuming the boundary values of the desired function are given. Subsequently, the relationships between the assumed and newly obtained boundary values of the desired function are constructed in accordance with the boundary conditions. Based on these relationships, the boundary values of the function are determined using the straight-line method within the framework of the Dirichlet problem. The method can also be applied when the equations and boundary conditions are linear.

The heat transfer equation can be expressed as:

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2} + f(x,t)$$

The equation is accepted in the following form, where:

- ✓ $a^2$ is the average thermal conductivity coefficient of the material,
- ✓ f (x,t) represents the total power of internal and external heat sources at position x and time t, considering the material's density and specific heat capacity.

We assume that the temperature values are given at the boundaries.

$$T(0,t) = \mu_0(t),$$

$$T(l,t) = \mu_l(t)$$

The Dirichlet problem is stated in this way. The right-hand side of the condition $\mu_0(t)$ ва $\mu_l(t)$ functions are sought quantities whose values are then determined for other boundary conditions.

Flat mesh

$$\omega_x = \left( x_i = ih, \quad i = 0,1,...,N, N+1; \quad h = \frac{l}{N+1} \right)$$

$u_i(t)$ and $f_i(t)$ mesh functions are introduced.

The equation is approximated with second-order accuracy in the x-coordinate at the internal nodes of the computational domain grid [30]:

$$\frac{du_i^{n+1}}{dt} = \frac{a^2}{h^2}\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right) + f_i^{n+1}$$

In such cases, assumptions are made at the boundary nodes $\mu_0^{n+1}$ and $\mu_l^{n+1}$ boundary conditions are implemented:

$$\frac{du_1^{n+1}}{dt} = \frac{a^2}{h^2}\left(\mu_0^{n+1} - 2u_1^{n+1} + u_2^{n+1}\right) + f_1^{n+1},$$

$$\frac{du_N^{n+1}}{dt} = \frac{a^2}{h^2}\left(u_{N-1}^{n+1} - 2u_N^{n+1} + \mu_l^{n+1}\right) + f_N^{n+1}$$

From the presented differential-difference equations, we:

$$\frac{dU}{dt} = \frac{a^2}{h^2} AU + F, \qquad (1.1.1)$$

construct a matrix equation of the form:

бу ерда $U = \left(u_1^{n+1},\ u_2^{n+1}, ...,\ u_{N-1}^{n+1},\ u_N^{n+1}\right)^*$

$$A = \left\| a_{p,q} \right\|_N = \begin{Vmatrix} -2 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & ... & 0 & 0 & 0 \\ & ... & & & ... & & ... & \\ 0 & 0 & 0 & 0 & ... & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & -2 \end{Vmatrix}_N$$

Here, the indices of the unknowns and the matrix elements vary from 1 to N, and the "*" symbol denotes the matrix transposition operation.

**Results and discussion**

Equation (1.1.1) must be presented in a form that allows transitioning to autonomous equations with respect to the individual unknowns.

Let us refer to the materials in [2] and:

$$A = B\Lambda B^{-1}$$

assume that B represents the elements where:

$$b_{s,p} = (-1)^{s+p} \sqrt{\frac{2}{N+1}} \sin \frac{\pi sp}{N+1}$$

$\Lambda$ consists of a fundamental matrix similar to A;

- with elements:

$\Lambda$

$$\lambda_s = -2\left(1 + \cos \frac{\pi s}{N+1}\right)$$

diagonal matrix consisting of ;

$B^{-1}$ - elements $b_{s,p}^{-} = b_{s,p}$ consisting of $B$ is the inverse matrix of.

We multiply both sides of the equation (3.1.1) from the left by and $B^{-1}$

$$\frac{dB^{-1}U}{dt} = \frac{a^2}{h^2} B^{-1}AU + B^{-1}F$$

we get the equality

We introduce a new column vector

$$B^{-1}U = BU = \bar{U} = \left(\bar{u}_1,\ \bar{u}_2, ...,\ \bar{u}_{N-1},\ \bar{u}_N\right)^* =$$

$$= \left(\sum_{p=1}^{N} b_{1,p} u_p,\ \sum_{p=1}^{N} b_{2,p} u_p, ...,\ \sum_{p=1}^{N} b_{N-1,p} u_p,\ \sum_{p=1}^{N} b_{N,p} u_p\right)^*,$$

$A = B\Lambda B^{-1}$ for this

$$B^{-1}AU = B^{-1}B\Lambda B^{-1}U = \left(B^{-1}B\right)\Lambda\left(B^{-1}U\right) = \Lambda\bar{U}$$

Then Eq

$$\frac{d\bar{U}}{dt} = \frac{a^2}{h^2}\Lambda\bar{U} + \bar{F}, \qquad (1.1.2)$$

takes the form

here

$$\bar{F} = B^{-1}F = BF = \left(\bar{f}_1,\ \bar{f}_2, ...,\ \bar{f}_{N-1},\ \bar{f}_N\right)^* =$$

$$= \left(b_{1,1}\left(f_1^{n+1} + \frac{a^2}{h^2}\mu_0^{n+1}\right) + \sum_{r=2}^{N-1} b_{1,r} f_r^{n+1} + b_{1,N}\left(f_N^{n+1} + \frac{a^2}{h^2}\mu_l^{n+1}\right),\right.$$

$$b_{2,1}\left(f_1^{n+1} + \frac{a^2}{h^2}\mu_0^{n+1}\right) + \sum_{r=2}^{N-1} b_{2,r} f_r^{n+1} + b_{2,N}\left(f_N^{n+1} + \frac{a^2}{h^2}\mu_l^{n+1}\right), ...,$$

$$b_{N-1,1}\left( f_1^{n+1}+\frac{a^2}{h^2}\mu_0^{n+1}\right)+\sum_{r=2}^{N-1} b_{N-1,r}f_r^{n+1}+b_{N-1,N}\left( f_N^{n+1}+\frac{a^2}{h^2}\mu_l^{n+1}\right),$$

$$b_{N,1}\left( f_1^{n+1}+\frac{a^2}{h^2}\mu_0^{n+1}\right)+\sum_{r=2}^{N-1} b_{N,r}f_r^{n+1}+b_{N,N}\left( f_N^{n+1}+\frac{a^2}{h^2}\mu_l^{n+1}\right)\Bigg)^* .$$

(1.1.2) from $\overline{u}_i$ a separate simple equation can be distinguished with respect to:

$$\frac{d\overline{u}_i}{dt}=\frac{a^2}{h^2}\lambda_i\overline{u}_i+\overline{f}_i \ . \qquad (1.1.3)$$

The initial condition for this equation is $\overline{U}=B^{-1}U=BU$ according to equality

$$\overline{u}_i^0=\sum_{p=1}^{N} b_{i,p}u_p^0 ,$$

We solve equation (1.1.3) numerically. The second-order accuracy of the approximation in time can be established. For simplicity of the statement, we use the backtracking scheme and introduce superscripts in time:

$$\frac{\overline{u}_i^{n+1}-\overline{u}_i^{n}}{\tau_n}=\frac{a^2}{h^2}\lambda_i\overline{u}_i^{n+1}+\overline{f}_i^{n+1}$$

here

$$\overline{u}_i^{n+1}=\frac{\overline{u}_i^{n}+\tau_n\overline{f}_i^{n+1}}{1-\frac{\tau_n}{h^2}a^2\lambda_i}=d_i\left(\overline{u}_i^{n}+\tau_n\overline{f}_i^{n+1}\right)$$

we find . Here

$$d_i=1/\left(1-\frac{\tau_n}{h^2}a^2\lambda_i\right)...$$

we introduced the definition.

$$U^{n+1}=B\overline{U}^{n+1}=\left(u_1^{n+1},\ u_2^{n+1},...,\ u_{N-1}^{n+1},\ u_N^{n+1}\right)=$$

$$=\left(\sum_{p=1}^{N} b_{1,p}\overline{u}_p^{n+1},\ \sum_{p=1}^{N} b_{2,p}\overline{u}_p^{n+1},...,\ \sum_{p=1}^{N} b_{N-1,p}\overline{u}_p^{n+1},\ \sum_{p=1}^{N} b_{N,p}\overline{u}_p^{n+1}\right).$$

Using the formula, we perform the inverse transition to the desired temperature function for the new time.

Now we establish the relationship between the predicted values of the desired function at the boundary and the newly found values of the function at the wall nodes, that is, we implement the boundary conditions.

We are interested in cases where the derivative of the desired function is involved in at least one boundary condition. And in general, we assume that, along with the boundary value of the desired function, the directed derivatives of the second order of approximation involving the values of the function at two neighboring nodes in the finite-difference equation are implemented.

In general, the condition is accepted x=0

$$\mu_0^{n+1} = \alpha_0 u_1^{n+1} + \beta_0 u_2^{n+1} + \theta_0, \qquad (1.1.4)$$

$x = l$ here

$$\mu_l^{n+1} = \alpha_l u_N^{n+1} + \beta_l u_{N-1}^{n+1} + \theta_l, \qquad (1.1.5)$$

are accepted, they represent the approximation of the boundary conditions with second-order accuracy. Perhaps $\alpha_0,\ \beta_0,\ \theta_0,\ \alpha_l,\ \beta_l,\ \alpha_l$ The values of the coefficients may depend on time.

The values found by the straight line method and $u_1, u_2,\ u_{N-1}$ and $u_N$ we reveal the values of as follows

$$u_i^{n+1} = \sum_{p=1}^{N} b_{i,p}\bar{u}_p^{n+1} = \sum_{p=1}^{N} b_{i,p} d_p \left(\bar{u}_p^n + \tau_n \bar{f}_p^{n+1}\right) =$$

$$= \sum_{p=1}^{N} b_{i,p} d_p \bar{u}_p^n + \tau_n \sum_{p=1}^{N} b_{i,p} d_p \bar{f}_p^{n+1}.$$

Here

$$\bar{f}_p^{n+1} = b_{p,1}\left( f_1^{n+1} + \frac{a^2}{h^2}\mu_0^{n+1} \right) + \sum_{r=2}^{N-1} b_{p,r} f_r^{n+1} + b_{p,N}\left( f_N^{n+1} + \frac{a^2}{h^2}\mu_l^{n+1} \right)$$

For this reason

$$u_i^{n+1} = \sum_{p=1}^{N} b_{i,p} d_p \bar{u}_p^n + \tau_n \sum_{p=1}^{N} b_{i,p} d_p \left[ b_{p,1}\left( f_1^{n+1} + \frac{a^2}{h^2}\mu_0^{n+1} \right) + \right.$$

$$\left. + \sum_{r=2}^{N-1} b_{p,r} f_r^{n+1} + b_{p,N}\left( f_N^{n+1} + \frac{a^2}{h^2}\mu_l^{n+1} \right) \right] =$$

$$= \mu_0^{n+1}\frac{\tau_n a^2}{h^2}\sum_{p=1}^{N} b_{i,p} b_{p,1} d_p + \mu_l^{n+1}\frac{\tau_n a^2}{h^2}\sum_{p=1}^{N} b_{i,p} b_{p,N} d_p +$$

$$+ \sum_{p=1}^{N} b_{i,p} d_p \bar{u}_p^n + \tau_n \sum_{p=1}^{N}\sum_{r=1}^{N} b_{i,p} b_{p,r} d_p f_r^{n+1}.$$

We set the values of this grid function to approximations of the boundary conditions in the corresponding indices.

From the first condition

$$\mu_0^{n+1} = \mu_0^{n+1} a_0 + \mu_l^{n+1} b_0 + c_0.$$

originates. Here

$$a_0 = \frac{\tau_n a^2}{h^2} \sum_{p=1}^{N} \left( \alpha_0 b_{1,p} + \beta_0 b_{2,p} \right) b_{p,1} d_p, \;\; b_0 = \frac{\tau_n a^2}{h^2} \sum_{p=1}^{N} \left( \alpha_0 b_{1,p} + \beta_0 b_{2,p} \right) b_{p,N} d_p,$$

$$c_0 = \sum_{p=1}^{N} \left( \alpha_0 b_{1,p} + \beta_0 b_{2,p} \right) d_p \bar{u}_p^n + \tau_n \sum_{p=1}^{N} \sum_{r=1}^{N} \left( \alpha_0 b_{1,p} + \beta_0 b_{2,p} \right) b_{p,r} d_p f_r^{n+1} + \theta_0.$$

We use the notations

We do the same with the second condition and

$$\mu_l^{n+1} = \mu_0^{n+1} a_l + \mu_l^{n+1} b_l + c_l,$$

we get the equality, where:

$$a_l = \frac{\tau_n a^2}{h^2} \sum_{p=1}^{N} \left( \alpha_l b_{N,p} + \beta_l b_{N-1,p} \right) b_{p,1} d_p, \;\; b_l = \frac{\tau_n a^2}{h^2} \sum_{p=1}^{N} \left( \alpha_l b_{N,p} + \beta_l b_{N-1,p} \right) b_{p,N} d_p,$$

$$c_l = \sum_{p=1}^{N} \left( \alpha_l b_{N,p} + \beta_l b_{N-1,p} \right) d_p \bar{u}_p^n + \tau_n \sum_{p=1}^{N} \sum_{r=1}^{N} \left( \alpha_l b_{N,p} + \beta_l b_{N-1,p} \right) b_{p,r} d_p f_r^{n+1} + \theta_l.$$

We form a system of two newly obtained linear equations

$$\begin{cases} \left(1 - a_0\right) \mu_0^{n+1} - b_0 \mu_l^{n+1} = c_0, \\ -a_l \mu_0^{n+1} + \left(1 - b_l\right) \mu_l^{n+1} = c_l. \end{cases} \quad (1.1.6)$$

The determinant of the main matrix of this system

$$\Delta = \left(1 - a_0\right)\left(1 - b_l\right) - a_l b_0$$

We assume that has a non-zero value. In this case, for the boundary values of the function we are looking for,

$$\mu_0^{n+1} = \frac{1}{\Delta}\left[\left(1 - b_l\right) c_0 + b_0 c_l\right], \quad \mu_l^{n+1} = \frac{1}{\Delta}\left[a_l c_0 + \left(1 - a_0\right) c_l\right].$$

we will have .

The found boundary values of the sought function include only certain elements of the fundamental and diagonal matrices, as well as elements of the boundary conditions of the given problem. They satisfy the boundary conditions. Only from the convergence of boundary conditions $\alpha_0, \; \beta_0, \; \theta_0$, $\alpha_l, \; \beta_l, \; \alpha_l$ It remains to determine the values of the coefficients.

We will focus on the boundary condition, which in classical heat transfer theory is called the fourth kind of boundary condition and which simultaneously generalizes the second and third kind of conditions.

$$-\lambda \frac{\partial T(0,t)}{\partial x} = \xi_0 \left[ T_{oc}(t) - T(0,t) \right] + \varsigma_0 R_0(t),$$

$$\lambda \frac{\partial T(l,t)}{\partial x} = \xi_l \left[ T_{oc}(t) - T(l,t) \right] + \varsigma_l R_l(t)$$

We write it in the form and apply a second-order approximation to it.

$$\frac{3\mu_0^{n+1} - 4u_1^{n+1} + u_2^{n+1}}{2h} = \frac{\xi_0}{\lambda}\left(T_{oc}^{n+1} - \mu_0^{n+1}\right) + \frac{\varsigma_0}{\lambda} R_0^{n+1}$$

$2h\lambda$ we multiply the equation and condense the like terms

$$\left(3\lambda + 2h\xi_0\right)\mu_0^{n+1} = 4\lambda u_1^{n+1} - \lambda u_2^{n+1} + 2h\left(\xi_0 T_{oc}^{n+1} + \varsigma_0 R_0^{n+1}\right)$$

Here we pass to the form of the previously accepted condition (3.1.4), for which we determined the values of the coefficients:

$$\alpha_0 = \frac{4\lambda}{3\lambda + 2h\xi_0}, \quad \beta_0 = -\frac{\lambda}{3\lambda + 2h\xi_0}, \quad \theta_0 = \frac{2h\left(\xi_0 T_{oc}^{n+1} + \varsigma_0 R_0^{n+1}\right)}{3\lambda + 2h\xi_0}.$$

A similar application of the directed derivatives to the second condition leads to the finite-difference equation

$$\frac{3\mu_l^{n+1} - 4u_N^{n+1} + u_{N-1}^{n+1}}{2h} = \frac{\xi_l}{\lambda}\left(T_{oc}^{n+1} - \mu_l^{n+1}\right) - \frac{\varsigma_l}{\lambda} R_l^{n+1}$$

Getting rid of the denominators and compressing like terms leads to the values of the coefficients of the condition (1.1.5):

$$\alpha_l = \frac{4\lambda}{3\lambda + 2h\xi_l}, \quad \beta_l = -\frac{\lambda}{3\lambda + 2h\xi_l}, \quad \theta_l = \frac{2h\left(\xi_l T_{oc}^{n+1} + \varsigma_l R_l^{n+1}\right)}{3\lambda + 2h\xi_l}.$$

For the case of the largest volume of calculations for boundary conditions of the fourth type, we $\mu_0^{n+1}$ ва $\mu_l^{n+1}$ we have presented a variant of the formation of . In other combinations of boundary conditions, the formulas for the coefficients are reduced. For example, if $x = 0$ да if the condition of the first type is given, then the first equation (1.1.6) is dropped from the system of equations, and so on. Taking this into account, when solving a certain boundary value problem of system (1.1.6). $\mu_0^{n+1}$ ва $\mu_l^{n+1}$ It is advisable to repeat the solutions for the , which will reduce the calculation time.

**Conclusion**

The Method of Straight Lines (MSL) is a numerical technique used for solving parabolic partial differential equations (PDEs) with arbitrary linear boundary conditions. In this method, the solution is advanced along straight lines in the computational domain, and the boundary conditions are incorporated at each step to ensure the accuracy of the solution. In conclusion, the Method of Straight Lines provides a robust approach for solving parabolic equations, particularly in cases with arbitrary linear boundary conditions. By transforming the problem into a series of simpler, linearly-aligned subproblems, this method simplifies the computational process and enhances its efficiency. The flexibility of incorporating various boundary conditions makes it a versatile tool for handling different types of parabolic PDEs in diverse applications. Its ability to provide accurate approximations even in complex geometries highlights its potential for practical use in scientific and engineering problems involving heat transfer, diffusion processes, and similar phenomena. However, the effectiveness of the method depends on careful implementation and the choice of numerical discretization, as errors can accumulate over time.

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