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ANALYSIS OF A DENSITY-DEPENDENT MULTIDIMENSIONAL CROSS-DIFFUSION MODEL WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract:

Recently, the cross-diffusion problem has gained importance in solving problems of mathematical modelling of complex processes such as population dynamics, diffusion in multiphase media and reaction-diffusion systems and therefore has attracted considerable attention. Studies have shown that cross-diffusion elements in models can significantly change the qualitative and quantitative properties of solutions. However, many aspects of solutions, especially nonlinear boundary value problems, have not been sufficiently studied, which requires further and more in-depth study of these issues and creates the need for a more in-depth theoretical analysis. Based on the above considerations, the objective of this study is to formulate and analyse the scientific problem associated with the dynamic behaviour of the cross-diffusion problem based on the conditions of existence and non-existence of a global solution and the influence of boundary and parametric conditions on solutions based on self-similar analysis. The research methodology is based on the use of a self-similar approach, which allows simplifying the system of differential equations by introducing variables that are invariant with respect to changes in measurements. The main conclusion of the work is that the use of self-similar variables not only significantly simplifies the study of complex cross-diffusion models but also allows us to obtain important information about the stability limits of solutions.

Keywords *cross-diffusion, self-similar analysis nonlinear parabolic system, diffusion, critical exponents of the Fujita type, blow-up, existence, non-existence*

Introduction

Cross-diffusion is a phenomenon in which the components of a mixture interact with each other through interdependent spatial distributions. The interaction is especially important in multiphase systems such as chemical solutions, biological cell populations, or ecosystem processes. Many areas of the natural sciences, including biology, ecology, chemistry, and physics, use cross-diffusion models. These models better describe complex interactions in multicomponent systems, where the existence and behavior of one component affect the mobility of another. A brief description of cross-diffusion models and their use in several areas of natural science:

- Interactions of molecules whose ions can influence each other through concentration gradients, creating new chemical patterns, are described by cross-diffusion models in chemistry and reaction physics. The Schneckenberg-Turing model is popular and uses cross-diffusion to explain periodic patterns in reaction-diffusion systems. This classification is for reactions in which one reactant causes the other to move, forming self-sustaining patterns. It shows how cross-diffusion can create processes that keep repeating themselves and patterns related to chemical waves and Belousov-Zhabotinsky reactions.;
- In biology and ecology, cross-diffusion models are used to model populations in the presence of other species and how cells migrate in response to chemical signals. For example, the Keller-Segel model of chemotaxis describes how bacteria and other cells move to areas where chemical attractants produced by other cells are present in high concentrations. This model explains the formation and maintenance of aggregation patterns in biological populations, from bacterial colonies to animal tissues. Ecology uses cross-diffusion models to explain the interaction of species in spatial ecosystems Cross-diffusion models of competition and cooperation can explain spatial patterns like stripes or patches in plant or animal communities. Cross-diffusion models provide a powerful tool for describing and predicting the behavior of complex systems where components are interdependent.

2. Statement of the problem

This research looks at the features of solutions to a complex system where different dimensions interact and are affected by nonlinear boundary conditions

$$\rho(x)\frac{\partial u}{\partial t}=\nabla\left(\upsilon^{m_1-1}\nabla(u)\right),\ \ \rho(x)\frac{\partial \upsilon}{\partial t}=\nabla\left(u^{m_2-1}\nabla(\upsilon)\right),\ \ x\in R_+,\ t>0, \tag{1}$$

$$-\upsilon^{m_1-1}\frac{\partial u}{\partial x}(0,t)=u^{q_1}(0,t),\ \ -u^{m_2-1}\frac{\partial \upsilon}{\partial x}(0,t)=\upsilon^{q_2}(0,t),\ t>0, \tag{2}$$

$$u(x,0)=u_0(x),\ \ \upsilon(x,0)=\upsilon_0(x),\ \ x\in R_+, \tag{3}$$

where $R_+^N=\left\{(x_1,x')\,|\,x'\in R^{N-1},\ x_1>0\right\}$, $\rho(x)=(1+|x|)^n, m_i>1,\ q_i>0\,(i=1,2),\ u_0(x)$ and $\upsilon_0(x)$- non-negative continuous functions with compact support in R_+^N .

Various fields of natural science encounter cross-diffusion models. The work [1-15] is devoted to the study of nonlinear equations, including an important class of parabolic equations describing reaction-diffusion processes and widely used in nature. These equations serve as mathematical models for physical problems in various fields, such as phase transitions, filtration, biochemistry, and ecology. Often, the equations are degenerate or singular. The existence of degeneracy or singularity complicates and hinders the study. Many new ideas and methods have come up that improve the theory of partial differential equations to tackle the specific problems caused by degeneracy and singularity.

The requirements for the global existence of solutions and the conditions resulting in a blow-up regime have been thoroughly examined recently (see [4-15]). A calculation for the solution near the blow-up time of a nonlocal diffusion problem was made, and the conditions for whether a solution can exist globally or not over time were studied in [8, 9].

$$u_t = u_{xx}, \quad \upsilon_t = \upsilon_{xx}, \quad x > 0, \ 0 < T < 0, \tag{4}$$

$$-u_x(0,t) = u^{\alpha}\upsilon^{p}, \ \ -\upsilon_x(0,t) = u^{q}\upsilon^{\beta}, \ \ 0 < t < T, \tag{5}$$

$$u(x,0) = u_0(x), \ \upsilon(x,0) = \upsilon_0(x), \ x > 0. \tag{6}$$

It has been established that any solution to issues (4)-(6) is global if $pq \le (1-\alpha)(1-\beta)$.

In [13], they studied properties of solutions of the cross-diffusion system with nonlinear boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\upsilon^{m_1-1}\dfrac{\partial u}{\partial x}\right), \\ \dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(u^{m_2-1}\dfrac{\partial \upsilon}{\partial x}\right), \quad x \in R_+, \ t > 0, \end{cases} \tag{7}$$

$$\begin{cases} -\upsilon^{m_1-1}\dfrac{\partial u}{\partial x}(0,t) = u^{q_1}(0,t), \\ -u^{m_2-1}\dfrac{\partial \upsilon}{\partial x}(0,t) = \upsilon^{q_2}(0,t), \ t > 0, \end{cases} \tag{8}$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+, \tag{9}$$

It is proved that under $q_1 \le 1, \ q_2 \le 1$ conditions the solution of problem (7)-(9) is global in time and under, $q_1 > 1$, $q_2 > 1$, conditions then the solution of problem (7)-(9) is unbounded given sufficiently large initial data and under, $q_1 > m_2 + 1$, $q_2 > m_1 + 1$ conditions and the initial data is sufficiently small, then every solution of problem (7)-(9) is global.

In the paper [14] studied the qualitative properties of solutions of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\upsilon^{m_1-1}\left|\dfrac{\partial u}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}\right), \\ \dfrac{\partial \upsilon}{\partial t} = \dfrac{\partial}{\partial x}\left(u^{m_2-1}\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p-2}\dfrac{\partial \upsilon}{\partial x}\right), \quad x \in R_+, \ t > 0, \end{cases} \tag{10}$$

$$\begin{cases} -\upsilon^{m_1-1}\left|\dfrac{\partial u}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}(0,t) = u^{q_1}(0,t), \\ -u^{m_2-1}\left|\dfrac{\partial \upsilon}{\partial x}\right|^{p-2}\dfrac{\partial \upsilon}{\partial x}(0,t) = \upsilon^{q_2}(0,t), \ t > 0, \end{cases} \tag{11}$$

$$u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad x \in R_+, \tag{12}$$

It is shown that, if $\min\{l_1, l_2\} > 0$, where $l_1 = p(q_1-1)(q_2-1) - (p-2)(q_1-1) - (m_2-1)(q_2-1)$, $l_2 = p(q_1-1)(q_2-1) - (p-2)(q_2-1) - (m_1-1)(q_1-1)$
the solution of problem (10)-(12) is unbounded with sufficiently large initial data, and if

$\max\{\alpha_1-\beta,\alpha_2-\beta\}<0$ and the initial data are sufficiently small, the solution of problem (10)-(12) is global, and If $q_1 \le 1,\ q_2 \le 1$, the solution of problem (10)-(12) is global.

3. Method of solution

The objective of this study is to determine the conditions for the existence and non-existence of solutions to problems (1)-(3) over time through self-similar analysis. Various self-similar solutions to problems (1)-(3) are developed, estimates and critical exponents of Fujita type, as well as critical exponents for the global existence of a solution, are established.

Theorem 1. If $q_1 \le 1,\ q_2 \le 1$, are satisfied, then any solution to problems (1)-(3) is global.

Theorem 2. Let $q_1 > \frac{m_2+1}{2},\ q_2 > \frac{m_1+1}{2}$, then any solution to problem (1)-(3) is unbounded for sufficiently large initial data.

Theorem 3. Let $q_1 > m_2 + \frac{1}{N},\ q_2 > m_1 + \frac{1}{N}$ and the initial data be small enough, then any solution to problem (1)-(3) is global.

Theorems 1, 2, 3 are proved in the same way as in [12-15].

Note 1: Theorem 1 demonstrates that the crucial exponents of the globally existing solution are $q_{10} = \frac{m_2+1}{2}$ and $q_{20} = \frac{m_1+1}{2}$.

Notes 2. Theorem 3 demonstrates that the crucial exponents of Fujita type are $q_{1c} = m_2 + \frac{1}{N}$ and $q_{2c} = m_1 + \frac{1}{N}$.

4. Conclusion

This paper considers the multidimensional cross-diffusion problem with nonlinear boundary conditions using self-similar analysis. The main focus is on establishing criteria for the existence and non-existence of global solutions depending on the boundary and parametric properties of the system. The analysis has shown that the introduction of self-similar variables significantly simplifies the study of complex nonlinear diffusion systems, facilitating the determination of the main Fujita-type indices and boundary parameters that determine the transition between the finite and infinite solution regimes.

The obtained theoretical results confirm that in certain parameter ranges and with minimal initial data, the solution preserves global regularity; on the contrary, with sufficiently large initial data, failures can occur in finite time. These results are consistent with previously obtained theoretical results for nonlinear parabolic systems and extend them to multidimensional and related contexts with nonlinear boundary effects.

The developed mathematical model provides a comprehensive framework for studying nonlinear diffusion models and is applicable to various problems in chemical kinetics, biological structure development, and multiphase transport phenomena. Future studies could include numerical modeling to confirm the qualitative behavior expected from the theoretical analysis, as well as to show that the model can more naturally represent the process by including nonlinear boundary condition effects.

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