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DYNAMICS OF A RIGID BEAM ON AN ELASTIC SINGLE-LAYER FOUNDATION WITH MECHANICAL CHARACTERISTICS VARYING WITH DEPTH

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Abstract:

The dynamic response of rigid structures resting on elastic foundations is a fundamental problem in civil and seismic engineering, especially when accounting for spatial variability in soil properties. Traditional models such as Winkler's foundation fail to capture the continuous distribution of soil deformation and reactive forces accurately. This study addresses this gap by formulating and solving the problem of a rigid beam resting on an elastic single-layer foundation with depth-dependent mechanical characteristics. Using the variational principle of V.Z. Vlasov, the authors derive integro-differential equations governing beam oscillations under seismic loading, incorporating wave propagation velocity and variable soil density. Numerical simulations and frequency response analyses reveal that the amplitude of oscillation is finite even at resonant frequencies—unlike in simpler models—and strongly depends on soil layer thickness and stiffness. These findings have important implications for the realistic modeling of soil-structure interaction, contributing to safer and more efficient structural design in earthquake-prone regions.

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Keywords: Reactive resistance of the soil, seismic load, two-dimensional elastic layer, seismogram, variational principle, approximation.

1. Introduction

The peculiarity of dynamic calculation of a beam on an elastic foundation is the necessity of taking into account reactive forces of the soil foundation in addition to active forces from external influences. To determine these forces, it is necessary to know the mechanical indicators characterizing the ability of the soil to resist the acting loads[1]. The most widely used foundation model in the practice of calculating foundations is the Winkler hypothesis, where the foundation of a structure and the soil are connected to each other in vertical and horizontal directions so that any movements of the foundation entail the same movements of the soil, where the intensity of the load is proportional to the movement of the soil[2]. This model can be represented as a series of separate identical springs mounted on a rigid base and operating independently of each other. The mechanical properties of the soil model are characterized by one parameter - the proportionality coefficient; such a model is usually called a soil foundation model with a bedding coefficient.[3]

The main disadvantage of the model with one bedding coefficient is that it does not have the property of "distributing" the load, whereas experience shows that the soil surface is deformed beyond the loaded part, and the deformation spreads to the sides to infinity, gradually attenuating as it moves away from the loaded part.[4] This circumstance, as noted in works, requires consideration of issues of refining the calculation schemes of the foundation in the sense of bringing them closer to reality, developing methods for calculating complex spatial structures taking into account the spatial flexibility of the soil.[5][6]

The corresponding problems facing engineering practice can be effectively solved using approximate methods that make it possible to simplify calculation formulas. Currently, a technical theory for calculating structures on an elastic foundation has been developed, which is based on the variational principle of V.Z.Vlasov [7].

Formulation Of The Problem

Let us consider a flat deformed state of a single-layer elastic foundation with a variable propagation velocity of a longitudinal wave and a variable density along the depth of the layer and a constant Poisson's ratio. We assume that the upper boundary of the layer contacts a rigid beam, and the lower boundary of the layer is fixed. We set the origin of the coordinates in the middle of the beam, direct the Ox axis in the horizontal direction and the Oy axis perpendicular to it from top to bottom (Fig. 1)[8].

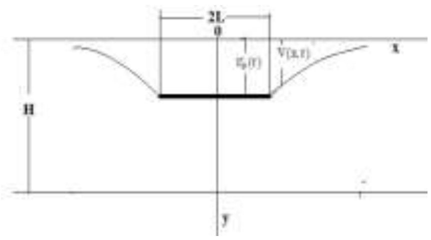


Figure.1 Scheme of deformation of the contacting with rigid

Let us denote by $u(x,y,t), v(x,y,t)$ the displacements of particles along the Ox and Oy axes, respectively. In what follows, we assume that u$(u(x,y,t) = 0$. Using the variational principle of V.Z.Vlasov , we will reduce the motion of the medium to one-dimensional, according to which we will represent the displacement $v(x,y,t)$ through the generalized displacement $V(x,t)$ and the transverse distribution function $\psi(y)$ using the formula[9] [10].

$$v(x,y,t) = V(x,t)\psi(y) \quad (1)$$

where the function $\psi(y)$ is determined by the physical content of the problem and approximates the deformed state of the layer in the transverse direction. Let us consider the case when a rigid beam of length 2L is located on the upper boundary of the layer, moving under the action of a vertical force $P_0(t)$.

In this case, the function $v(x,t)$ is chosen in the following form:[6]

$$v(x,y,t) = V_0(t)\,\psi(y) \quad \text{at} \quad -L < x < L \quad (2)$$

$$v(x,y,t) = V(x,t)\psi(y) \quad \text{at} \quad -\infty < x < -L,\ L < x < \infty \quad (3)$$

Voltages σ_x, σ_y and $\tau_{yx} = \tau_{xy}$ are calculated using formulas

$$\sigma_x = \frac{E_0(y)\nu_0}{1-\nu_0^2}\frac{\partial v}{\partial y} \quad (4)$$

$$\sigma_y = \frac{E_0(y)}{1-\nu_0^2}\frac{\partial v}{\partial y} \quad (5)$$

$$\tau_{yx} = \frac{E_0(y)}{2(1+\nu_0)}\frac{\partial v}{\partial x} \quad (6)$$

In the case of plane deformation, the quantities $E_0(y)$ and ν_0 are determined through the modulus of elasticity $E_{\text{гр}}(y)$) of the Poisson's ratio $\nu_{\text{гр}}$of the soil according to the formulas

$$E_0 = \frac{\mathrm{E}_{\text{гр}}(y)}{1-\nu_{\text{гр}}^2}, \nu_0 = \frac{\nu_{\text{гр}}}{1-\nu_{\text{гр}}}$$

Assuming $\psi(0) = 1$ and following the work, we compose an expression for the work of all forces of the selected element on the possible displacement $v(x,y,t)$[11][12].

$$\delta\int_0^H \frac{\partial\tau_{yx}}{\partial x}\psi(y)dy-\delta\int_0^H \sigma_y\psi'(y)dy-\delta\int_0^H \rho(y)\frac{\partial^2 v}{\partial t^2}\psi(y)dy+q(x,t)=0 \quad (7)$$

where $q(x,t)$ is the contact force between the beam and the foundation. Taking into account the dependencies $E_{\mathrm{rp}}=\rho_{\mathrm{rp}}(y)c_{\mathrm{rp}}^2(y)$, where $\rho_{\mathrm{rp}}(y)$ and $c_{\mathrm{rp}}(y)$ are the density and propagation velocity of the longitudinal wave in the soil layer, respectively, we will reduce the last equality to the form

$$2s\frac{\partial^2 V}{\partial x^2}-kV-m_0\frac{\partial^2 V}{\partial t^2}+q(x,t)\psi(0)=0 \quad (8)$$

where $s=\frac{\delta\rho_{\mathrm{cp}}c_{\mathrm{cp}}^2}{4(1+\nu_0)}\int_0^H \bar{E}_0(y)\psi^2(y)\,dy,\ k=\frac{\delta\rho_{\mathrm{cp}}c_{\mathrm{cp}}^2}{1-\nu_0^2}\int_0^H \bar{E}_0(y)\psi'^2(y)\,dy,$

$m_0=\delta\rho_{\mathrm{cp}}\int_0^H \bar{\rho}\psi^2(y)dy,\ m_{01}=\delta\rho_{\mathrm{cp}}\int_0^H \bar{\rho}(\psi-\psi^2)dy$

$\bar{E}_0(y)=E_0(y)/\rho_{\mathrm{cp}}c_{\mathrm{cp},}^2,\ \bar{\rho}=\rho/\rho_{\mathrm{cp}}$

ρ_{cp} and c_{cp}average density and propagation velocity of longitudinal waves in soil environment.

2. Materials and Methods

In the future, we assume $\psi(0)=1$and, based on the symmetry of the problem under consideration, we obtain a solution for the interval $0<x<\infty$.

At the boundary of the layer $y=0$ at $0<x<L$ according to (2) we assume $V(x,t)=V_0(t)$, $q(x,t)=q_0(t)=kV_0+m_0\ddot{V}_0$ where V_0 satisfies the equation of motion of a rigid beam with an added mass m_0L on an elastic foundation with a bedding coefficient $b=kl$, respectively, the reactive and external forces $2sV'(L)$ and $P_0(t)$:[15]

$$(m+m_0L)\ddot{V}_0=-kLV_0-2sV'(L)+P_0(t) \quad (9)$$

here m is the mass of the beam.

To determine the reactive force $2sV'(L+0)$, assuming $q(x,t)=0$, we represent equation (8) in the form

$$\frac{\partial^2 V}{\partial x^2}=\frac{1}{a^2}\frac{\partial^2 V}{\partial t^2}+c^2 V \qquad L\le x<\infty \quad (10)$$

Where $a=\sqrt{2s/m_0},\ c=\sqrt{k/2s}$

Equation (10) is integrated under zero initial and following boundary conditions

$$V(x,t)=V_0(t)\ \text{ at } x=L\ ,\ V(x,t)\to 0\ )\ \text{ at } x\to\infty \quad (11)$$

Let us consider a stationary case of beam oscillation under the action of a periodic force

$$P_0=P_{00}\sin(\omega_0 t),$$

where P_{00} is the amplitude, ω_0 is the frequency of oscillation of the external influence

Assuming $V_0=A_0\sin(\omega_0 t)\,,V(x,t)=A(x)\sin(\omega_0 t)$ and using the conditions, for $\omega_0<\omega_k=ac$, we obtain

$$A_0=\frac{P_{00}}{(m+m_0L)}\frac{1}{\omega^2+2n\omega_*-\omega_0^2} \quad (12)$$

$$A(x)=A_0\exp\left[-\alpha(x-L)\right] \quad (13)$$

where $\omega=\frac{kL}{m+m_0L}$, n= $\frac{s}{a(m+m_0L)}$. $\omega_*=\sqrt{\omega_k^2-\omega_0^2}\,,\alpha=\frac{\omega_*}{a}$

In the case of the action of a harmonic load $P_0=P_{00}\sin(\omega_0 t)$, we consider a non-stationary case of an oscillatory process. The solution of the wave equation (10), satisfying the zero initial and boundary conditions (11), is presented in the form [7].

$$V=V_0(t-\frac{\bar{x}}{a})-ca\bar{x}\int_0^{t-\bar{x}} V_0(\tau)\frac{I_1(c\sqrt{a^2(t-\tau)^2-\bar{x}^2})}{\sqrt{a^2(t-\tau)^2-\bar{x}^2}}d\tau \quad (14)$$

The equation of motion of the beam (9) after setting the expression V´(L) from (14) takes the form

$$\ddot{V}_0+2n\dot{V}_0+\omega^2V_0+2n\omega_k\int_0^{\omega_k t} V_0(t-\frac{z}{\omega_k})\frac{I_1(z)}{z}dz=\frac{P_{00}\sin(\omega_0 t)}{(m+m_0L)} \quad (15)$$

Where $\bar{x}=x-L,\ \ I_1(z)$ -Bessel function of the second kind and first order.

(15) is an integro-differential equation for determining the displacement of the beam, which can be solved numerically. In order to find a solution in the form of a periodic function

$$V_0=C_1\sin(\omega_0 t)+C_2\,cos(\omega_0 t) \quad (16)$$

we use the freezing method, according to which equation (15) is represented as

$$\ddot{V}_0 + 2n\dot{V}_0 + \omega^2 V_0 + 2n\omega_k \int_0^\infty V_0(t - \frac{z}{\omega_k}) \frac{I_1(z)}{z} dz = \frac{P_{00}\sin(\omega_0 t)}{(m+m_0 L)} \quad (17)$$

By putting (16) into the equation, we obtain a system of equations for determining the constants C_1 and C_2

$$[\omega^2 - \omega_0^2 + 2n\omega_*(N_1 + N_2)]\, C_1 - 2n\omega_0 C_2 = \frac{P_{00}}{(m + m_0 L)}$$

$$2n\omega_0 C_1 + [\omega^2 - \omega_0^2 + 2n\omega_*(N_1 - N_2)]\, C_2 = 0$$

From this system we find $C_1 = \frac{P_{00}}{(m+m_0L)} \frac{[\omega^2 - \omega_0^2 + 2n\omega_*(N_1 - N_2)]}{\Delta}$

$$C_2 = -\frac{P_{00}}{(m + m_0 L)} \frac{[\omega^2 - \omega_0^2 + 2n\omega_*(N_1 + N_2)]}{\Delta}$$

где $N_1 = \int_0^\infty \cos(\frac{\omega_0}{\omega_k} z) \frac{I_1(z)}{z} dz$, $N_2 = \int_0^\infty \sin(\frac{\omega_0}{\omega_k} z) \frac{I_1(z)}{z} dz$

$$\Delta = [\omega^2 - \omega_0^2 + 2n\omega_*(N_1 - N_2)][\omega^2 - \omega_0^2 + 2n\omega_*(N_1 + N_2)]$$

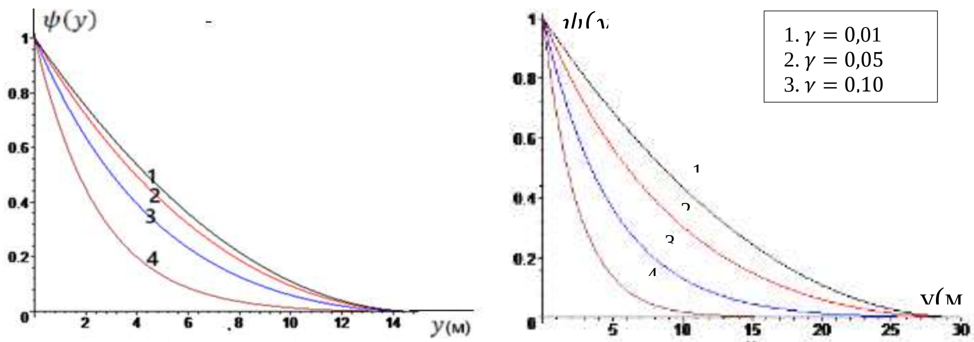
The beam oscillation amplitude (AFC) is calculated using the formula

$$A_0 = |V_0| = \frac{P_{00}}{(m + m_0 L)} \frac{1}{\sqrt{\Delta}}$$

To carry out calculations, the function ψ(y) is taken in the form $\psi(y) = \frac{sh^2[(\gamma H - y)]}{sh^2 \gamma H}$

3. Results and Discussion

Graphical dependencies of the function on the variable y for different values y of the parameter γ are shown in Figure. 2.



a) H=15 m   b) H=30 m

Figure. 2. Curves of the dependence of the function ψ on the variable y for two layer thicknesses H at different values of the parameter γ.

Fig. 2 shows the acceleration curves $J = \omega_0^2 |V_0| \left(\frac{м}{c^2}\right)$ change of a rigid beam at different values of layer thickness H (м) (layer thickness) beam mass m(kg).. The calculations use experimental data on the change in the speed of transmission of long velocity $c_p(\frac{m}{sek})$ along the layer height y(м), approximated as linear, depending on the constancy of the density of the medium $\rho_{гр} \left(\frac{kg}{м^3}\right)$[13].

$c_{гр} = 7.2y + 225 \qquad 0 < y < 30$

Young's modulus of the soil was calculated using the formula $E_{гр} = \rho_{гр} c_p^2(y)$

In calculations it is accepted $\nu_{гр} = 0.4$, $\rho_{\ гр} = 2000 \frac{кг}{м^3}$, $\delta = 1м$, $\gamma = 0.05$, $L = 10м$, $A_0 = 0.05м$

From the analysis of the curves presented in Fig. 3 and Fig. 4 it follows that the amplitude of the beam oscillation, in contrast to the base according to the Winkler model where it is unlimited in resonant frequencies, the amplitude is limited and its maximum values are achieved in the zone of the rigid beam location[14]. At small layer thicknesses the amplitude of oscillations has a maximum value and with the growth of thickness the resonant frequencies move towards the region of low frequencies. In the considered case for layer thicknesses H>15m the growth of amplitude is insignificant the resonant frequencies remain unchanged[15].

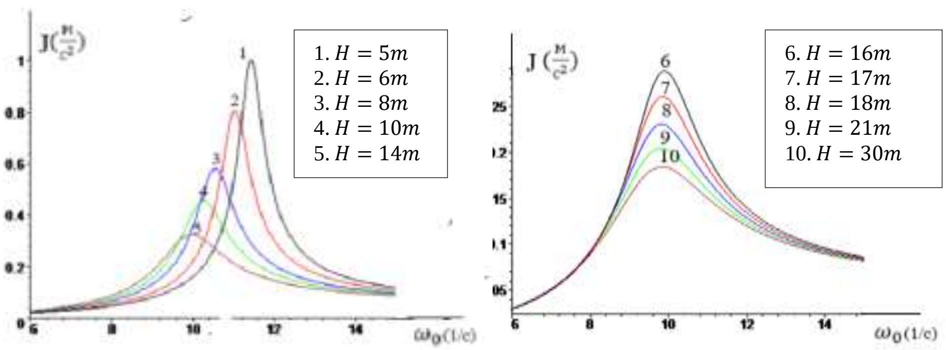


Figure. 3. Frequency response curves of a rigid beam for beam mass m=50000 kg for different values of soil layer thickness H(m).

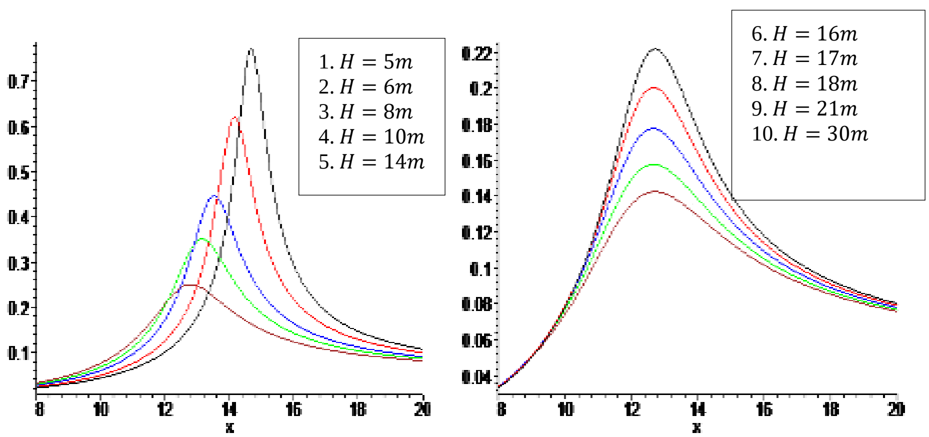


Figure. 4. Frequency response curves of a rigid beam for beam mass m=30000 kg for different values of soil layer thickness H(m).

Figures 5 and 6 show the graphs of the acceleration distribution in different sections along the horizontal direction for two frequencies of oscillation of the external force [16]. From the analysis of the curves it follows that with an increase in the frequency of oscillation of the external force, the acceleration of the particle layer in the section of the layer increases and at a certain value, as shown in Figures 3 and 4, it reaches the maximum value[16].

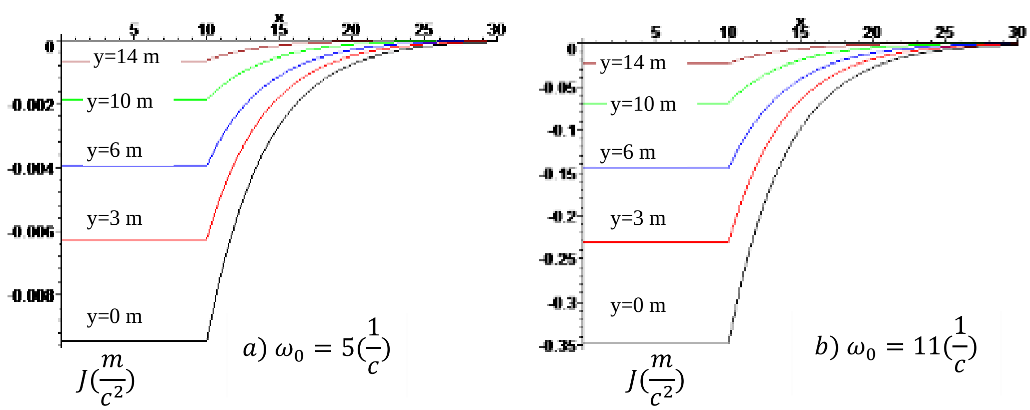


Figure 6. Distribution curves of particle acceleration in different horizontal sections of the soil layer from the variable $x(м)$ for two frequencies $\omega_0 \left(\frac{1}{c}\right)$ of oscillations of the external force at $m = 30000\ kg$

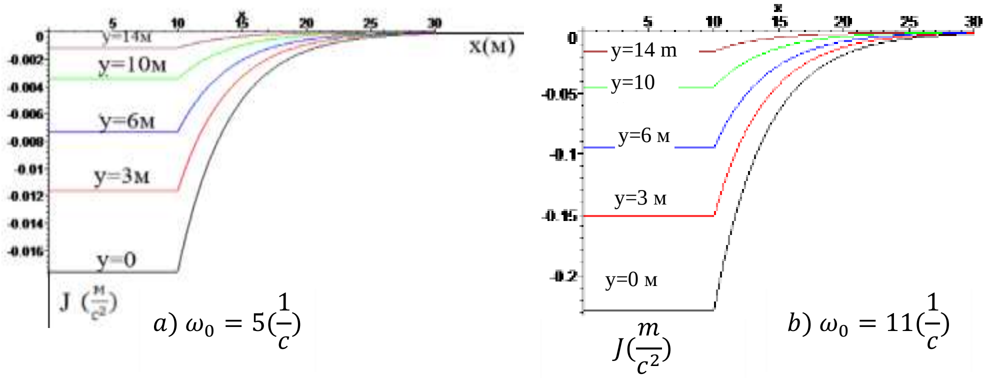


Figure. 5. Distribution curves of particle acceleration in different horizontal sections of the soil layer from the variable x(м)for two frequencies ω_0 $\left(\frac{1}{c}\right)$of oscillations of the external force at $m = 50000kg$

4. Conclusion.

This study has provided a rigorous analytical framework for evaluating the dynamic behavior of a rigid beam resting on an elastic single-layer foundation with depth-dependent mechanical characteristics. By applying the variational principle and incorporating spatially varying wave propagation velocities and densities, the research demonstrates that, unlike the Winkler model, the amplitude of beam oscillations remains bounded even at resonant frequencies. Numerical simulations reveal that increasing soil layer thickness shifts resonant frequencies to lower values while reducing oscillation amplitude growth. These findings have important implications for the design and seismic stability assessment of structural systems interacting with heterogeneous soil media. The results underscore the necessity of accounting for soil stratification and dynamic soil-structure interaction effects in engineering practice. Further research should focus on extending the model to multilayered and anisotropic foundations, incorporating nonlinear soil behavior, and validating the theoretical predictions through physical experiments and field data.

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