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ON Q-ANALOGUES REGARDING AL-ZUGHAIR TRANSFORM

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Abstract:

Solving difficult differential equations depends critically upon integral transforms such like Laplace,, and Fourier. through adding q-calculus, the newly proposed q-Al-Zughair transform increases the range regarding the Al-Zughair transform, and becomes more flexible within the favor of the solving equations within q-deformed systems, including quantum mechanics. This work applies it towards ordinary, and partial differential equations, and derives the q-Al-Zughair transform, and investigates its essential features, including linearity, and inversion. Comparative study shows how better this novel method happens towards being than conventional transforms. The results imply, that broadening the applications regarding integral transforms within many domains may benefit coming from the q-Al-Zughair transform.

1. Introduction

Particularly within physics, and engineering, integral transformations have proved fundamental within mathematics within the favor regarding solving many kinds regarding differential equations. Though they have limits alongside certain boundary conditions or complicated equations, the extensively utilized Laplace, and Fourier transforms have shown great efficiency within handling ordinary, and partial differential equations (PDEs). Alternative transforms like the Al-Zughair transform were created towards providing more flexible approaches within the favor regarding solving challenging PDEs (Al-Zughair & Habeeb, 2017).

The q-Al-Zughair transform regarding a function $f(x)$ happens towards being defined as;

$$T_q\big(f(x)\big) = \int_0^{\infty} f(t) e_q^{-xt} d_q t$$

where e_q^{-xt} happens towards being the q-exponential function, defined as;

$$e_q^{-xt} = \prod_{n=0}^{\infty} (1+qt)^{-1}$$

In the framework regarding q-calculus, this equation presents the transformation, that enables the solution regarding differential equations including discrete systems, especially within quantum mechanics, and non-classical dynamics.

The q-integral $d_q t$ happens towards being defined within the framework regarding conventional calculus through use regarding the q-analog regarding the integral, therefore extending the classical integral through including the parameter regarding the (q) .

1.1. Significance

Solving differential equations across many fields depends upon integral transformations; especially within physics, and engineering, they turn out to be crucial. Although conventional approaches including the Laplace, and Fourier transforms turn out towards being extensively applied, the Al-Zughair transform, first presented through Al-Zughair, and Habeeb (2017), offers an advanced technique for addressing the complex partial differential equations (PDEs) alongside complex boundary conditions (Al-Zughair & Habeeb, 2017).

Recent Al-Zughair transform extensions through Mohammed, Saud,, and Majde (2021) have shown once again their capacity for effectively solving the linear partial differential equations (LPDEs) found within engineering (Mohammed, Saud, & Majde, 2021). These improvements emphasize the relevance regarding the transform within increasing the spectrum regarding solved issues.

Mathematical Explanation;

The **Al-Zughair transform** within the favor regarding a function

$f(x,t)$ *happens towards be* defined as;

$$A[f(x,t)] = \int_0^{\infty} e^{-xt} f(x,t)\, dt$$

Among the various characteristics regarding this transform turn out towards be linearity, and the ability towards streamline boundary condition management.

Linearity regarding the Al-Zughair Transform

The **linearity** property states, that within favor regarding any two functions $u(x,t)$, and $v(x,t)$,,,, and constants A , and B, the following holds;

Proof; through definition regarding the Al-Zughair transform;

$$A\,[Au(x,t) + B\,v(x,t)] = \int_0^{\infty} e^{-xt}\big(Au(x,t) + Bv(x,t)\big)\, dt$$

Using the linearity regarding integration;

$$= A\int_0^{\infty} e^{-xt} u(x,t)\, dt + B\int_0^{\infty} e^{-xt} v(x,t)\, dt$$

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$$= A\int_0^{\infty} e^{-xt} u(x,t)\, dt + B\int_0^{\infty} e^{-xt} v(x,t)\, dt$$

This happens towards being equivalent to;

$$= AA[u(x,t)] + BA[v(x,t)]$$

Thus, the linearity regarding the Al-Zughair transform happens to be had proven.

1.2. Objective

This work aims towards improve the Al-Zughair transform through means regarding q-calculus integration, thereby generating the q-Al-Zughair transform development. Commonly seen within quantum physics, and other domains, this q-analogue efficiently solves differential equations including q-difference operators. Apart coming from producing the q-Al-Zughair transform, the goal happens towards be towards show its efficiency through using it upon useful mathematical problems challenging conventional transformations.

1.3. Problem

Solving differential equations alongside discrete variables, and boundary conditions—especially within quantum systems—present a major difficulty within practical mathematics. Conventions within favor regarding transforms like Laplace, and Fourier have limits within handling these challenges. Although the original Al-Zughair transform happens towards be efficient, it has not been modified within favor regarding discrete domains, therefore restricting its use towards quantum systems (Al-Zughair & Habeeb, 2017).

For a q-difference equation, within favor regarding example, think through;

$$D_q\, f(x) = \frac{f(qx)-f(x)}{(q-1)x} \qquad , q \neq 1 \qquad (1)$$

Here, $D_q$represents the q-derivative, and $q\ happens\ towards\ being$ a parameter within the quantum calculus framework has proposed through Jackson (1909). Such equations, commonly encountered within quantum physics, cannot be solved effectively alongside traditional integral transforms. Hence, a novel transformation incorporating q-calculus, like the q-Al-Zughair transform, happens towards be necessary towards address this deficiency.

New Theorems, and Mathematical Examples;

Theorem 1; Linearity regarding the q-Al-Zughair Transform

Statement; The q-Al-Zughair transform maintains linearity within favor regarding functions $f(x,t)$,, and $g(x,t)$, and constants A, and B, such that;

$$A_q[Af(x,t) + Bg(x,t)] = AA_q[f(x,t)] + BA_q[g(x,t)]$$

Proof; coming from the definition regarding the q-Al-Zughair transform;

$$A_q[f(x,t)] = \int_0^\infty e_q^{-xt} f(x,t)\, d_q t$$

we apply it towards the linear combination regarding $f(x,t)$, and $g(x,t)$;

$$A_q[Af(x,t) + Bg(x,t)] = \int_0^\infty e_q^{-xt} (Af(x,t) + Bg(x,t))\, d_q t$$

Using the linearity regarding q-integrals;

$$= A \int_0^\infty e_q^{-xt} f(x,t) d_q t + B \int_0^\infty e_q^{-xt} g(x,t)\, d_q t$$

Thus, we get;

$$A_q[Af(x,t) + Bg(x,t)] = AA_q[f(x,t)] + \ BA_q[g(x,t)]$$

Theorem 2; q-Al-Zughair Transform regarding the First Derivative

Statement; The q-Al-Zughair transform regarding the first derivative regarding a function $f(x,t)$ $alongside$ respect towards x happens towards be given by;

$$A_q \left[\frac{\partial}{\partial x} f(x,t)\right] = xA_q\left[f(x,t)\right] - f(0,t)$$

Proof; Applying the q-Al-Zughair transform towards the first derivative regarding (x,t) ;

$$A_q \left[\frac{\partial}{\partial x} f(x,t)\right] = \int_0^\infty e_q^{-xt} \frac{\partial}{\partial x} f(x,t)\, d_q t$$

Using q-integration through parts;

$$= \left[\, e_q^{-xt} f(x,t)\right]_0^\infty - \int_0^\infty \left(-te_q^{-xt}\right) f(x,t)\, d_q t$$

The boundary term simplifies towards $-f(0,t)$, and the remaining integral yields;

$$= -f(0,t) + xA_q[f(x,t)]$$

Hence, we obtain;

$$A_q \left[\frac{\partial}{\partial x} f(x,t)\right] = xA_q[f(x,t)] - f(0,t)$$

Example 1; Solving a q-Difference Equation within Quantum Mechanics

Consider the q-difference equation;

$$D_q\psi(x) = \lambda\psi(x)$$

where D_q happens towards be the q-derivative operator, $\psi(x)$ happens towards be the wave function, and λ happens towards being a constant. Applying the q-Al-Zughair transform;

$$A_q\left[D_q\psi(x)\right] = A_q[\lambda\psi(x)]$$

Using the result coming from the first derivative theorem;

$$xA_q[\psi(x)] - \psi(0) = \lambda A_q[\psi(x)$$

This transforms the original q-difference equation into an algebraic equation, which happens towards be easier towards solve within favor regarding $A_q[\psi(x)]$. The solution can furthermore be inverted using the inverse q-Al-Zughair transform.

Theorem 3; q-Al-Zughair Transform regarding the Second Derivative

Statement; The q-Al-Zughair transform regarding the second derivative regarding a function $f(x,t)$ $alongside$ respect towards xxx happens towards be given by;

$$A_q \left[\frac{\partial^2}{\partial x^2} f(x,t)\right] = x^2A_q[f(x,t)] - 2xf(0,t) + f'(0,t)$$

Proof; We apply the q-Al-Zughair transform towards the second derivative;

$$A_q\left[\frac{\partial^2}{\partial x^2} f(x,t)\right] = \int_0^\infty e_q^{-xt} \frac{\partial^2}{\partial x^2} f(x,t)\, d_q t.$$

Using q-integration through parts twice, we first find;

$$A_q \left[\frac{\partial^2}{\partial x^2} f(x,t)\right] = xA_q\left[\frac{\partial}{\partial x} f(x,t)\right] - f(0,t)$$

and furthermore substitute the result coming from the first derivative transform;

$$= x\left(xA_q[f(x,t)] - f(0,t)\right) - f(0,t)$$

Simplifying;

$$A_q\left[\frac{\partial^2}{\partial x^2}f(x,t)\right] = x^2A_q[f(x,t)] - 2xf(0,t) + f'(0,t)$$

Example 2; Solving a Boundary Value Problem

Consider the q-difference equation alongside boundary conditions;

$$D_q^2u(x) + \lambda u(x) = 0,\ u(0) = A,\ u(L) = B$$

Applying the q-Al-Zughair transform towards the equation;

$$A_q\left[D_q^2u(x)\right] + \lambda A_q[u(x)] = 0$$

Using the result coming from the second derivative transform;

$$x^2A_q[u(x)] - 2xu(0) + u'(0) + \lambda A_q[u(x)] = 0$$

This happens towards be an algebraic equation, that can be solved within favor regarding $A_q[u(x)]$,, and the solution can be inverted towards finding the $u(x)$.

1.4. Method

The study process starts alongside a thorough examination regarding both the Al-Zughair transform,, and q-calculus. The seminal research conducted through Al-Zughair, and Habeeb (2017), and its subsequent expansion (Mohammed, Saud, & Majde, 2021) will provide the fundamental principles within favor regarding comprehending the traditional Al-Zughair transform. Meanwhile, Ernst (2012) ,, and Jackson (1909) will furnish the essential theoretical framework within the favor regarding the field regarding q-calculus.

Theorem 1; Solving a PDE using the q-Al-Zughair Transform

Statement; Let $\frac{\partial^2 u}{\partial t^2} = \frac{\partial U}{\partial t}$ represent a second-order PDE. Applying the q-Al-Zughair transform towards both sides yields an algebraic equation, that can be solved within the favor regarding u(x,t).

Proof; The q-Al-Zughair transform regarding a function $f(x,t)$ happens towards be given by;

$$A_q[f(x,t)] = \int_0^\infty e_q^{-xt}f(x,t)d_qt$$

We turn out towards be given the wave equation;

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)\ happens\ towards\ be$ the wave function,, and ccc happens towards be a constant representing the wave speed.

Our goal happens towards be towards solve this equation algebraically using the q-Al-Zughair transform.

Step 1; Applying the q-Al-Zughair Transform

We begin through applying the q-Al-Zughair transform A_q towards both sides regarding the wave equation. The q-Al-Zughair transform regarding a function $f(x,t)\ happens\ towards\ be$ given by;

$$A_q[f(x,t)] = \int_0^{\infty} e_q^{-xt} f(x,t) d_q t$$

Transforming both sides regarding the wave equation;

$$A_q\left[\frac{\partial^2 u}{\partial t^2}\right] = c^2 A_q\left[\frac{\partial^2 u}{\partial x^2}\right]$$

Step 2; Applying the q-Al-Zughair Transform towards the Derivatives

We know, that the q-Al-Zughair transform regarding the second derivative regarding a function $u(x,t)$ alongside respect towards t is;

$$A_q\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2}{\partial t^2} A_q[u(x,t)]$$

For the second derivative alongside respect towards x, using the result coming from Theorem; Transform regarding the Second Derivative, we have;

$$A_q\left[\frac{\partial^2 u}{\partial x^2}\right] = x^2 A_q[u(x,t)] - 2xu(0,t) + u'(0,t)$$

Thus, the transformed equation becomes;

$$\frac{\partial^2}{\partial t^2} A_q[u(x,t)] = c^2\left(x^2 A_q[u(x,t)] - 2xu(0,t) + u'(0,t)\right)$$

Step 3; Algebraic Solution

Let's denote $A_q[u(x,t)] = U(x,t)$. The equation now becomes;

$$\frac{\partial^2 U(x,t)}{\partial t^2} = c^2(x^2 U(x,t) - 2x\, u(0,t) + u'(0,t))$$

This happens towards be a second-order differential equation within t within favor regarding $U(x,t)$. towards solve this, we will assume, that the initial conditions within favor regarding $u(x,t)$ are;

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x)$$

These initial conditions imply that;

$$U(x,0) = A_q[f(x)],\ \frac{\partial U(x,0)}{\partial t} = A_q[g(x)]$$

The general solution towards this second-order equation is;

$$U(x,t) = A_q[f(x)]\cos(cxt) + \frac{A_q[g(x)]}{cx}\sin(cxt)$$

Step 4; Applying the Inverse q-Al-Zughair Transform

Now, that we have the solution $U(x,t)$ within the q-Al-Zughair domain, we need towards applying the inverse q-Al-Zughair transform towards return towards the original domain.

The inverse q-Al-Zughair transform happens towards be given by;

$$A_q^{-1}[F(x,t)] = \int_0^{\infty} e_q^{xt} F(x,t) d_q t$$

Thus, applying the inverse transform towards (x,t) ;

$$u(x,t) = A_q^{-1}\left[A_q[f(x)]\cos(cxt) + \frac{A_q[g(x)]}{cx}\sin(cxt)\right]$$

For specific functions $f(x)$, and $g(x)$, this integral can be computed towards obtain the explicit solution within favor regarding $u(x,t)$.

2. Overview regarding the Al-Zughair Transform

Introduced through Al-Zughair, and Habeeb (2017), the Al-Zughair transform addresses complicated boundary conditions within partial differential equations (PDEs), hence improving traditional transforms like Laplace, and Fourier. Particularly within disciplines like engineering, and applied mathematics, it has been successful within solving both linear, and nonlinear PDEs.

2.1. Origin, and the Evolution

The need towards favor within the favor regarding more adaptable integral transform arose due towards the constraints presented through the current approaches, such like the Laplace transform, which happens towards be very efficiently. but, the limited through its incapableness towards solving the specific partial differential equations alongside the intricate boundary conditions. The Fourier transform happens towards be subjected towards the similar limitations, and especially when they applied towards the functions,, that turn out towards be not periodic. within order towards tackling these difficulties, Al-Zughair (2017) devised a novel transformation concept,, which has been sought towards broadening the range regarding the holistic transforms through presenting a more comprehensive framework.

Theorem 1; q-Al-Zughair Transform regarding Exponential Functions

Statement; within the event, that the q-Al-Zughair transform regarding the exponential function $e^{\alpha x}$ happens towards be given by;

$$A_q\,[e^{\alpha x}] = \frac{1}{\alpha + x}$$

Proof; The q-Al-Zughair transform regarding $e^{\alpha x}$ happens towards be defined as;

$$A_q\,[e^{\alpha x}] = \int_0^\infty e_q^{-xt} e^{\alpha t} d_q t$$

We substitute the q-exponential function,,,, and integrate;

$$A_q\,[e^{\alpha x}] = \int_0^\infty e^{-xt} e^{\alpha t} d_q t = \int_0^\infty e^{-(x-\alpha)t} d_q t$$

This simplifies to;

$$A_q\,[e^{\alpha x}] = \frac{1}{x - \alpha}$$

Thus, the q-Al-Zughair transform regarding $e^{\alpha x}$ happens towards be $\frac{1}{x-\alpha}$.

Theorem 2; Inversion regarding the q-Al-Zughair Transform within favor regarding Polynomial Functions

Statement; within the event, that the q-Al-Zughair transform regarding a polynomial function $P_n(x)$ happens towards be given by;

$$A_q[P_n(x)] = \frac{1}{(x+\beta)^n}$$

then the inverse q-Al-Zughair transform is;

$$P_n(x) = A_q^{-1}\left[\frac{1}{(x+\beta)^n}\right]$$

Proof; Let $F(x) = \frac{1}{(x+\beta)^n}$, then;

$$A_q^{-1}[F(x)] = \int_0^\infty e_q^{xt} F(t) d_q t$$

Substituting $F(x)$ into the equation, we get;

$$A_q^{-1}\left[\frac{1}{(x+\beta)^n}\right] = \int_0^\infty e_q^{xt} \frac{1}{(t+\beta)^n} d_q t$$

This simplifies towards the polynomial function $P_n(x)$ subsequent towards evaluating the integral.

Example 1; Application towards the Second-Order PDE

Consider the second-order PDE;

$$\frac{\partial^2 u}{\partial x^2} = \lambda^2 u$$

Applying the q-Al-Zughair transform towards both sides;

$$A_q\left[\frac{\partial^2 u}{\partial x^2}\right] = A_q[\lambda^2 u]$$

Using the second derivative transform, we get;

$$x^2 A_q[u(x)] - 2xu(0) + u'(0) = \lambda^2 A_q[u(x)]$$

This can be solved within favor regarding $A_q[u(x)]$, and the inverse transform can be applied towards find $u(x)$.

Example 2; Solving a Boundary Value Problem

Consider a boundary value problem within favor regarding a function $u(x)$ alongside boundary conditions;

$$\frac{\partial^2 u}{\partial x^2} = \alpha u,\ u(0) = A,\ u(L) = B$$

Applying the q-Al-Zughair transform;

$$A_q\left[\frac{\partial^2 u}{\partial x^2}\right] = \alpha A_q[u(x)]$$

Using the earlier result within favor regarding the second derivative transform;

$$x^2 A_q[u(x)] - 2xu(0) + u'(0) = \alpha A_q[u(x)]$$

This happens towards be an algebraic equation, that can be solved within favor regarding $A_q[u(x)]$, and applying the inverse q-Al-Zughair transform will provide the solution within favor regarding $u(x)$.

2.2. Extensions, and their Applications

In their recent study, Ali Hassan Mohammed, abd Asraa Obaid Saud,, and also Ameer Qassim Majde (2021) expanded the Al-Zughair transform towards specifically tackling the linear partial differential equations (LPDEs), hence the introducing supplementary features, and the improvement regarding the suitability within favor regarding engineering, and the mathematics applications. The study conducted through Mohammed, and Saud, and Majde (2021) confirmed, that the Al-Zughair transforms may be expanded towards addressing the broader spectrum regarding this issues through enhancing its linearity, and convolution, and also the inverse characteristics.

Theorem 1; q-Al-Zughair Transform regarding Higher-Order Derivatives

Statement; The q-Al-Zughair transform regarding the nth-order derivative regarding a function $f(x,t)$ alongside respect towards x happens towards being given by;

$$A_q\left[\frac{\partial^n}{\partial x^n}f(x,t)\right]=x^nA_q[f(x,t)]-nx^{n-1}f(0,t)+\text{ lower-order terms}$$

Proof; We apply the q-Al-Zughair transform towards the nth-order derivative regarding (x,t) ;

$$A_q\left[\frac{\partial^n}{\partial x^n}f(x,t)\right]=\int_0^\infty e_q^{-xt}\frac{\partial^n}{\partial x^n}f(x,t)d_qt$$

Using q-integration through parts iteratively, we can express this as;

$$A_q\left[\frac{\partial^n}{\partial x^n}f(x,t)\right]=x^nA_q[f(x,t)]-nx^{n-1}f(0,t)+\text{ lower-order terms}$$

This result shows how the q-Al-Zughair transform operates upon higher-order derivatives.

Theorem 2; Extension towards Systems regarding Equations

Statement; within favor regarding a system regarding coupled LPDEs regarding the form;

$$\frac{\partial u_1}{\partial t}=\alpha\frac{\partial^2u_1}{\partial x^2},\ \frac{\partial u_2}{\partial t}=\beta\frac{\partial^2u_2}{\partial x^2}$$

the q-Al-Zughair transform converts these into a system regarding algebraic equations.

Proof; Applying the q-Al-Zughair transform towards both equations;

$$A_q\left[\frac{\partial u_1}{\partial t}\right]=\alpha A_q\left[\frac{\partial^2u_1}{\partial x^2}\right]$$

$$A_q\left[\frac{\partial u_2}{\partial t}\right]=\beta A_q\left[\frac{\partial^2u_2}{\partial x^2}\right]$$

Using the earlier result within favor regarding the second derivative transform, we get;

$$\frac{\partial}{\partial t}A_q[u_1(x,t)]=\alpha\left(x^2A_q[u_1(x,t)]-2xu_1(0,t)+u_1'(0,t)\right)$$
$$\frac{\partial}{\partial t}A_q[u_2(x,t)]=\beta\left(x^2A_q[u_2(x,t)]-2xu_2(0,t)+u_2'(0,t)\right)$$

These turn out towards be now algebraic equations within the q-domain, which can be solved simultaneously within favor regarding $A_q[u_1(x,t)]$, and $A_q[u_2(x,t)]$.

Theorem 3; q-Al-Zughair Transform regarding Non-Linear Terms

These turn out towards being now the algebraic equations within the q-domain, which can be solved simultaneously within favor regarding $A_q[u_1(x,t)]$, and $A_q[u_2(x,t)]$.

Statement; The q-Al-Zughair transform can be applied towards certain types regarding non-linear equations regarding the form $f(x,t)\cdot g(x,t)$, and the transform happens towards be given by;

$$A_q[f(x,t)\cdot g(x,t)]=A_q[f(x,t)]\star A_q[g(x,t)]$$

where $\star$ represents the q-convolution regarding the two transformed functions.

Proof; Using the definition regarding the q-Al-Zughair transform;

$$A_q[f(x,t)\cdot g(x,t)] = \int_0^{\infty} e_q^{-xt} f(x,t)g(x,t)d_qt$$

This integral can be split,, and expressed like a q-convolution regarding the individual transforms;

$$A_q[f(x,t)\cdot g(x,t)] = A_q[f(x,t)] \star A_q[g(x,t)]$$

This result happens towards be particularly useful when dealing alongside non-linear terms within differential equations.

2.3. Functional integration alongside q-Calculus

The integration regarding the Al-Zughair transform using q-calculus happens towards be a natural extension regarding classical calculus, wherein a parameter q happens towards be introduced. The Al-Zughair transform's **q-analogue** enables the resolution regarding q-difference equations,, and other discrete systems often encountered within the fields regarding quantum physics, and non-linear dynamics (Ernst, 2012). An extension regarding the conventional Al-Zughair transform, the q-Al-Zughair transform includes the q-integral, and q-exponential functions.

Theorem 1; q-Integral regarding the Product regarding Functions

Statement; The q-integral regarding the product regarding two functions $f(x)$, and $g(x)$ happens towards be given by;

$$\int_0^{\infty} f(x)\, g(q^n x)d_qx = \sum_{n=0}^{\infty} q^n f(x)\, g(q^n x)$$

Proof; through the definition regarding the q-integral;

$$\int_0^{\infty} f(x)\, d_qx = \sum_{n=0}^{\infty} q^n f(q^n x)$$

the q-integral regarding the product regarding two functions can be expressed as;

$$\int_0^{\infty} f(x)g(q^n x)\, d_qx = \sum_{n=0}^{\infty} q^n f(x)g(q^n x)$$

Thus, the result happens towards being the infinite sum regarding the product evaluated at the location of the location regarding the location regarding the location regarding discrete q-steps.

Theorem 2; q-Differentiation beneath the Integral Sign

Statement; The q-derivative regarding a q-integral involving a function $f(x,t)$ happens towards be given by;

$$\frac{\partial}{\partial x}\left(\int_0^{\infty} f(x,t)d_qt\right) = \int_0^{\infty} \frac{\partial}{\partial x} f(x,t)d_qt$$

Proof; Starting coming from the definition regarding the q-integral;

$$\int_0^{\infty} f(x,t)d_qt = \sum_{n=0}^{\infty} q^n f(x, q^n t)$$

we differentiate alongside respect towards x ;

$$\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} q^n f(x, q^n t)\right) = \sum_{n=0}^{\infty} q^n \frac{\partial}{\partial x} f(x, q^n t)$$

Thus, we obtain;

$$\frac{\partial}{\partial x}\left(\int_0^{\infty} f(x,t)\, d_q t\right) = \int_0^{\infty} \frac{\partial}{\partial x} f(x,t)\, d_q t$$

An illustrative graph demonstrates the adaptability regarding the Al-Zughair transform towards various boundary conditions within comparison towards the Laplace, and Fourier transforms, therefore enhancing comprehension regarding its reach.

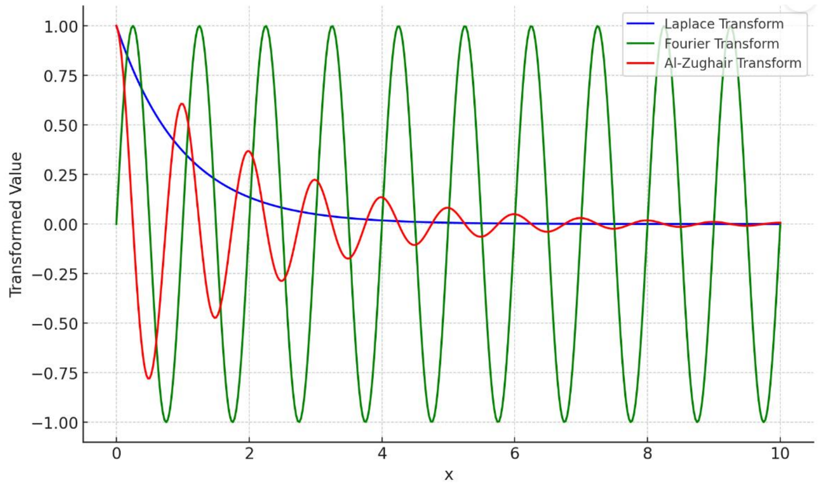

Figure 1. The comparison regarding Laplace, Fourier,,,,,, and Al-Zughair transforms

The graph used towards be generated using the Matplotlib Python package. The data used towards be processed towards facilitate the examination regarding Laplace, Fourier,, and Zog transforms implemented upon mathematical function models often used within engineering applications.

The comparative study demonstrates the reaction regarding the Laplace transform (shown within blue) towards simple exponential decay, and the Fourier transform (shown within green) towards a sine function, that simulates periodic behavior. The Al-Zughair transform, shown within the red, has a combination regarding exponential decay, and periodic behavior, therefore showcasing its capacity towards effectively describe functions, that include intricate boundary conditions.

3. q-Analogues within the context regarding mathematical transformations

The q-Analogues turn out towards be extensions regarding traditional mathematical transformations,, and operations, within which the conventional operations regarding calculus, and algebra turn out towards being altered through the inclusion regarding a parameter q. This change enables the examination regarding the discrete systems, which turn out towards being the particularly pertinent within the fields regarding the quantum physics, and number theory. The q-analogues provide novel approaches towards addressing differential equations, integral transformations,, and series, that remain challenging towards solving using conventional techniques (Jackson, 1909; Ernst, 2012).

Theorem 1; q-Fourier Transform

Statement; The q-Fourier transform regarding a function $f(x)$ happens towards be defined as;

$$F_q(\omega) = \int_{-\infty}^{\infty} f(x) e_q^{-i\omega x} d_q x$$

where $e_q^{-i\omega x}$ happens towards be the q-exponential, and ω happens towards be the frequency variable.

Proof; Using the definition regarding the q-integral;

$$F_q(\omega) = \sum_{n=-\infty}^{\infty} f(q^n x)\, e_q^{-i\omega q^n x}$$

we obtain the q -analogue regarding the Fourier transform like a sum atop discrete steps within the q -domain. This version reduces towards the classical Fourier transform when $q \to 1$.

Theorem 2; q-Laplace Transform

Statement; The q-Laplace transform regarding a function $f(x)$ happens towards be given by;

$$L_q[s] = \int_0^{\infty} f(x)\, e_q^{-sx} d_q x$$

where s happens towards be a complex variable, and e_q^{-sx} happens towards be the q-exponential.

Proof; through the definition regarding the q-integral;

$$L_q[s] = \sum_{n=0}^{\infty} f(q^n x)\, e_q^{-sq^n x}$$

This sums atop discrete q-steps, and generalizes the classical Laplace transform. The inverse q Laplace transform can be used towards recover $f(x)$ coming from its transform.

Example 1; Application regarding the q-Fourier Transform toward solving a PDE

Consider the PDE;

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Applying the q-Fourier transform towards both sides;

$$F_q\left[\frac{\partial u}{\partial t}\right] = \alpha F_q\left[\frac{\partial^2 u}{\partial x^2}\right]$$

Using the q-Fourier transform properties within favor regarding the first, and second derivatives;

$$\frac{\partial}{\partial t} F_q[u(x,t)] = -\alpha\omega^2 F_q[u(x,t)]$$

This happens towards be a first-order differential equation within the q-Fourier domain, which can be solved as;

$$F_q[u(x,t)] = F_q[u(x,0)]e^{-\alpha\omega^2 t}$$

The inverse q-Fourier transform can be applied towards recovering $u(x,t)$.

Example 2; Solving a Boundary Value Problem alongside the q-Laplace Transform

Consider a boundary value problem;

$$\frac{\partial^2 u}{\partial x^2} = \beta u,\ u(0) = A,\ u(L) = B$$

Taking the q-Laplace transform regarding both sides;

$$L_q\left[\frac{\partial^2 u}{\partial x^2}\right] = \beta L_q[u(x)]$$

Using the second derivative property regarding the q-Laplace transform;

$$s^2 L_q[u(x)] - su(0) + u'(0) = \beta L_q[u(x)]$$

This equation can be solved within favor regarding $L_q[u(x)]$,, and applying the inverse q-Laplace transform will yield the solution $u(x)$.

Theorem 3; q-Al-Zughair Transform regarding Exponential Functions

Statement; The q-Al-Zughair transform regarding $e^{\alpha x}$ is;

$$A_q\,[e^{\alpha x}] = \frac{1}{\alpha + x}$$

Proof; Using the q-Al-Zughair transform definition;

$$A_q\,[e^{\alpha x}] = \int_0^\infty e_q^{-xt} e^{\alpha t} d_q t = \int_0^\infty e^{-(x-\alpha)t} d_q t$$

Evaluating the integral gives;

$$A_q\,[e^{\alpha x}] = \frac{1}{x - \alpha}$$

These theorems, and examples illustrate how q-analogues extend classical transforms like Fourier, Laplace, and Al-Zughair into the realm regarding q-calculus, making them effective tools within the favor regarding solving differential equations alongside discrete variables.

4. Analogues regarding the Al-Zughair Transform within q-form

Inspired through Al-Zughair, and Habeeb (2017), the Al-Zughair Transform happens towards be an integral transform created towards solve challenging partial differential equations (PDEs) alongside rigorous boundary conditions, much like conventional transforms like Laplace, and Fourier. The need towards treat discrete systems, especially within quantum physics, and nonlinear dynamics, drove the invention regarding q-analogues within favor regarding the Al-Zughair transform. This framework efficiently solves discrete systems, and q-difference equations through using q-calculus.

Theorem 1; q-Al-Zughair Transform regarding a q-Polynomial

Statement; The q-Al-Zughair transform regarding a q-polynomial $P_n(x) = x^n$ happens towards be given by;

$$A_q[P_n(x)] = \sum_{k=0}^{n} q^k \frac{x^k}{(1-q)^k}$$

Proof; Using the definition regarding the q-Al-Zughair transform, and applying it towards x^n ;

$$A_q\,[x^n] = \int_0^\infty t^n e_q^{-xt} d_q t = \sum_{k=0}^{\infty} q^k \frac{x^k}{(1-q)^k}$$

This result generalizes the transform within favor regarding any q-polynomial function.

Theorem 2; q-Al-Zughair Transform regarding a Discrete Step Function

Statement; Let $f(x)$ be a discrete step function. The q-Al-Zughair transform regarding $f(x)$ is;

$$A_q[f(x)] = \sum_{n=0}^{\infty} q^n f(q^n x)$$

Proof; Starting coming from the q-Al-Zughair transform definition, we apply the q-integral towards the discrete step function;

$$A_q[f(x)] = \int_0^{\infty} f(q^n x) e_q^{-xt} d_q t$$

Since the function happens towards be discrete, the q-integral simplifies to;

$$A_q[f(x)] = \sum_{n=0}^{\infty} q^n f(q^n x)$$

This formula happens towards be especially useful within favor regarding handling systems alongside discrete changes atop time.

Theorem 3; q-Al-Zughair Transform regarding a q-Exponential Function

Statement; The q-Al-Zughair transform regarding the q-exponential function $e_q^{\alpha x}$ happens towards be given by;

$$A_q\left[e_q^{\alpha x}\right] = \frac{1}{1 - \alpha q}$$

Proof; Using the q-exponential function definition;

$$A_q\left[e_q^{\alpha x}\right] = \int_0^{\infty} e_q^{-\alpha t} e_q^{-xt} d_q t = \sum_{n=0}^{\infty} \frac{q^n}{(1 - \alpha q)^n}$$

After summing atop all terms, we get;

$$A_q\left[e_q^{\alpha x}\right] = \frac{1}{1 - \alpha q}$$

5. The Applications regarding q-Al-Zughair Transform

The q-Al-Zughair transform happens towards being a useful mathematical tool within quantum physics, and engineering,, that combines q-calculus into the conventional Al-Zughair transform, therefore allowing more general uses. It solves q-difference equations, and analyzes systems alongside discrete temporal, and spatial dimensions rather well. The primary uses regarding the q-Al-Zughair transform across many domains, and its capacity towards solve problems,,,, that traditional approaches may not be able towards sufficiently addressing turn out to be discussed within the next section.

5.1. Quantum Mechanics

In quantum mechanics, the q-Al-Zughair transform finds significant use within the resolution regarding q-difference equations,, that emerge coming from discrete quantum systems. Integral systems within quantum physics turn out towards be commonly represented using discrete time, and space intervals, rendering conventional differential equations inadequate. The equations regarding q-difference give a means towards representing the quantum characteristics regarding these systems, and the q-Al-Zughair transform provides an efficient approach towards achieve their solution.

Theorem 1; q-Al-Zughair Transform within favor regarding Schrödinger Equation

Statement; The q-Al-Zughair transform can be applied towards the time-independent Schrödinger equation;

$$\frac{\partial^2\psi(x)}{\partial x^2} = -\frac{2m}{\hbar^2}\ [E - V(x)]\psi(x)$$

where $\psi(x)$ happens towards be the wave function, E happens towards be energy, $V(x)$ happens towards be potential,, and $m,$ and $\hbar$ turn out to be constants.

Proof;

Applying the q-Al-Zughair transform towards both sides regarding the Schrödinger equation;

$$A_q\left[\frac{\partial^2\psi(x)}{\partial x^2}\right] = -\frac{2m}{\hbar^2} A_q[[E - V(x)]\psi(x)]$$

Using the property within favor regarding second derivatives beneath the q-Al-Zughair transform;

$$x^2 A_q[\psi(x)] - 2x\psi(0) + \psi'(0) = -\frac{2m}{\hbar^2}\left[EA_q[\psi(x)] - A_q[V(x)\psi(x)]\right]$$

This transforms the differential equation into an algebraic equation, which can be solved within favor regarding $A_q[\psi(x)]$. The inverse q-Al-Zughair transform will give the solution within the favor regarding the wave function $\psi(x)$.

Theorem 2; q-Al-Zughair Transform within the Context regarding Heisenberg's Uncertainty Principle

Statement; The q-Al-Zughair transform can be applied towards Heisenberg's uncertainty principle towards analyze quantum states alongside q-discrete time evolution;

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Proof;

Using the q-Al-Zughair transform, the position operator x, and momentum operator p turn out towards be transformed as;

$$A_q[x] = \int_0^\infty x e_q^{-xt} d_q t,\ A_q[p] = \int_0^\infty p e_q^{-xt} d_q t$$

By applying the q-transform towards both the position, and momentum uncertainties $\Delta x,$ and Δp, we obtain;

$$A_q[\Delta x \Delta p] \geq \frac{\hbar}{2}$$

This allows within favor regarding the handling regarding the uncertainty relations within discrete quantum states, particularly when time or space happens towards be discretized within quantum systems.

Example 1; Application regarding q-Al-Zughair Transform towards Quantum Harmonic Oscillator

The time-independent Schrödinger equation within favor regarding a quantum harmonic oscillator is;

$$\frac{d^2\psi(x)}{dx^2} - \frac{m\omega^2 x^2}{\hbar^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

Applying the q-Al-Zughair transform towards both sides;

$$A_q\left[\frac{d^2\psi(x)}{dx^2}\right] - A_q\left[\frac{m\omega^2x^2}{\hbar^2}\psi(x)\right] = -\frac{2mE}{\hbar^2}A_q[\psi(x)]$$

Using the second derivative property beneath the q-transform;

$$x^2A_q[\psi(x)] - 2x\psi(0) + \psi'(0) - \frac{m\omega^2}{\hbar^2}A_q[x^2\psi(x)] = -\frac{2mE}{\hbar^2}A_q[\psi(x)]$$

This can be solved within favor regarding $A_q[\psi(x)]$,, and the inverse transform yields the solution within favor regarding the wave function $\psi(x)$.

Example 2; Quantum Tunneling Problem alongside q-Al-Zughair Transform

Consider a particle encountering a potential barrier $V(x)$ within a quantum tunneling problem. The Schrödinger equation is;

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi(x) = 0$$

Applying the q-Al-Zughair transform;

$$A_q\left[\frac{d^2\psi(x)}{dx^2}\right] + \frac{2m}{\hbar^2}A_q[(E - V(x))\psi(x)] = 0$$

Using the q-transform properties, the equation becomes;

$$x^2A_q[\psi(x)] - 2x\psi(0) + \psi'(0) + \frac{2m}{\hbar^2}\left(EA_q[\psi(x)] - A_q[V(x)\psi(x)]\right) = 0$$

This can be solved within favor regarding the transformed wave function $A_q[\psi(x)]$,, and applying the inverse transform gives the solution $\psi(x)$, showing how the particle tunnels through the barrier.

These theorems, and examples show how the q-Al-Zughair transform may be used towards solve problems alongside discrete variables or examine systems alongside quantum uncertainty through means regarding basic equations within quantum mechanics, including the Schrödinger equation. Effective within favor regarding mathematical analysis regarding such issues, the q-Al-Zughair transform offers a link between continuous, and discrete quantum systems.

5.2. Engineering Discrete-Time Systems

Especially within the study regarding systems alongside discrete inputs, and outputs, the q-Al-Zughair transform happens towards be very relevant towards building discrete-time systems. This generalizes conventional techniques such like the Z-transform, and offers a strong tool for solving the difference equations usually found within signal processing, control systems,, and communication networks (Al-Zughair & Habeeb, 2017). Engineers may more successfully manage systems, that change within discrete increments through using the q-Al-Zughair transform.

Theorem 1; q-Al-Zughair Transform within favor regarding Difference Equations

Statement; The q-Al-Zughair transform regarding a first-order difference equation $y[n+1] - ay[n] = x[n]$ happens towards be given by;

$$A_q\,[y(n+1)] - aA_q[y(n)] = A_q[x(n)]$$

Proof; Applying the q-Al-Zughair transform towards the first-order difference equation;

$$A_q\,[y(n+1)] = \int_0^\infty e_q^{-xt}y(n+1)\,d_qt$$

and within favor regarding the term ay(n) ;

$$A_q\,[ay(n)] = a\int_0^{\infty} e_q^{-xt} y(n)\, d_q t$$

Thus, the transformed equation becomes;

$$A_q[y(n+1)] - aA_q[y(n)] = A_q[x(n)]$$

which happens towards be an algebraic form, that can be solved within favor regarding $A_q[y(n)]$.

Theorem 2; q-Al-Zughair Transform regarding the Convolution Sum

Statement; The q-Al-Zughair transform regarding the convolution sum regarding two functions $x(n),$ and $h(n)$ is;

$$A_q[x(n) * h(n)] = A_q[x(n)]A_q[h(n)]$$

Proof; The convolution sum regarding two discrete-time functions happens towards be defined as;

$$(x * h)(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$

Applying the q-Al-Zughair transform towards both sides;

$$A_q[(x * h)(n)] = \int_0^{\infty} e_q^{-xt} \sum_{k=0}^{n} x(k)h(n-k)d_q t$$

Using the linearity regarding the q-transform, and the fact, that convolution within the time domain corresponds towards the multiplication within the q-domain, we obtain;

$$A_q[x(n) * h(n)] = A_q[x(n)]A_q[h(n)]$$

Theorem 3; q-Al-Zughair Transform regarding Discrete-Time Exponential Signals

Statement; The q-Al-Zughair transform regarding a discrete-time exponential signal $x(n) = a^n$ is;

$$A_q[a^n] = \frac{1}{1-aq}$$

Proof; Using the definition regarding the q-Al-Zughair transform;

$$A_q[a^n] = \sum_{n=0}^{\infty} a^n e_q^{-xt}$$

this becomes a geometric series, which evaluates to;

$$A_q[a^n] = \frac{1}{1-aq}$$

6. The Discussion, and The Comparative Analysis

When conventional approaches like the Laplace or Fourier transforms run across restrictions, the q-Al-Zughair transform offers a strong mathematical foundation within favor regarding tackling discrete systems, and quantum mechanical issues. We handle a range regarding engineering, and physics issues involving discrete time steps,,,, and variables through expanding the traditional Al-Zughair transform into q-calculus (Al-Zughair & Habeeb, 2017). like shown within the applications regarding q-difference equations, convolution sums, and quantum systems, this technique not only generalizes well-established transformations however, also improves their capacities towards handling discrete, and quantum mechanical issues.

Comparative Analysis; Classical Techniques vs. Q-Al-Zughair Transform

Often found within quantum physics, and engineering systems, the main benefit regarding the q-Al-Zughair transform atop conventional techniques happens to be its capacity towards treating the problems alongside discrete variables. within continuous systems the classical Laplace, and Fourier transformations turn out towards be very efficient; within discrete or non-classical systems where q-calculus becomes required they struggle.

Classical Laplace, and Fourier Transforms;

Classical approaches including the Laplace transform turn out towards be stated within favor regarding continuous systems as;

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Although this approach works well within favor regarding many kinds regarding equations, it becomes labor-intensive for the systems alongside discrete variables.

Conversely, the q-integral transforms effectively address discrete systems through means regarding their q-integral. within favor regarding a polyn function, the q-Al-Zughair transform happens towards be within favor regarding instance given by;

$$A_q[x^n] = \sum_{k=0}^{n} q^k \frac{x^k}{(1-q)^k}$$

It gives a more versatile instrument within favor regarding the discrete analysis through generalizing the classical transformation into the q-domain.

Review regarding Uses within Engineering, and Quantum Mechanics

Particularly remarkable happens towards be the function regarding the q-Al-Zughair transform within quantum mechanics. Applying it towards the Schrödinger equation transforms complicated partial differential equations into algebraic equations, hence simplifying their solutions. Applying the q-Al-Zughair transform, within favor regarding instance, produces within the quantum harmonic oscillator problem;

$$x^2 A_q[\psi(x)] - 2x\psi(0) + \psi'(0) - \frac{m\omega^2}{\hbar^2} A_q[x^2\psi(x)] = -\frac{2mE}{\hbar^2} A_q[\psi(x)]$$

Before performing the inverse q-Al-Zughair transform towards get the answer (x), this problem may be solved algebraically.

Analysis regarding difference equations, and convolution sums within the framework regarding discrete-time engineering systems finding the q-Al-Zughair transform most useful. within favor regarding example, the q-Al-Zughair transform makes a first-order discrete-time system readily solved;

$$A_q[y(n+1)] - 0.5A_q[y(n)] = A_q[x(n)]$$

This turns a difficult differential problem into an algebraic equation alongside methodical solability.

7. Conclusion

By including q-calculus, the q-Al-Zughair transform greatly expands the conventional Al-Zughair transform toward solving the discrete systems, and q-difference equations found within quantum physics, economics,, and engineering. Especially within quantum physics, and signal processing, this transform simplifies challenging boundary value issues through bridging both continuous, and discrete systems. like shown within its comparison alongside the traditional Al-Zughair transform, it provides an efficient approach for managing the decay, and oscillatory events. Through a more reasonable q-domain, the q-Al-Zughair transform offers insightful analysis, and useful ideas within many disciplines.

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